ELECTROMAGNETIC CONIC SECTIONS

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Abstract

Many different languages are used to describe vector calculus, resulting in a "vector calculus gap" between mathematicians and other scientists. After first comparing several different traditional approaches to defining the derivative operators div, grad, and curl in curvilinear coordinates, we present a new approach based on "electromagnetic conic sections".

1 Introduction

What are the divergence and curl of a vector field? Students in mathematics courses often learn algebraic formulas for these derivatives, without learning the geometry behind them. Those students who go on to take physics or engineering courses which use these concepts often have trouble "bridging the gap" between the way vector calculus is taught by mathematicians and the way it is used in applications [1].

After summarizing several different ways to compute the divergence and curl, we present a new approach which combines geometry and physics. Using "electromagnetic conic sections", we show how to define these operators in several important cases. Not only is our approach tied to a physical interpretation in terms of the electromagnetic field, it is also a useful way to remember the formulas themselves.

2 Calculating Div, Grad, and Curl

2.1 Mathematics

In introductory mathematics courses, one typically works in Cartesian coordinates, using the basis $\{\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}\}$. Given any vector field

$$\vec{\boldsymbol{F}} = P\,\hat{\boldsymbol{\imath}} + Q\,\hat{\boldsymbol{\jmath}} + R\,\hat{\boldsymbol{k}} \tag{1}$$

the divergence and curl of $\vec{\boldsymbol{F}}$ are defined by the formulas

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{F}} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \tag{2}$$

and

$$\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{F}} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \tag{3}$$

respectively. It is in instructive exercise to try to use these formulas to verify that

$$\vec{\boldsymbol{\nabla}} \cdot \frac{(x\,\hat{\boldsymbol{\imath}} + y\,\hat{\boldsymbol{\jmath}} + z\,\hat{\boldsymbol{k}})}{(x^2+y^2+z^2)^{3/2}} = 0 \tag{4}$$

(the charge density for a point charge, away from the source) and that

$$\vec{\boldsymbol{\nabla}} \times \frac{(x\,\hat{\boldsymbol{\jmath}} - y\,\hat{\boldsymbol{\imath}})}{(x^2+y^2)} = \vec{\boldsymbol{0}} \tag{5}$$

(the current density of a line charge, away from the source). These are perhaps the 2 most important elementary physical examples of divergence and curl.

2.2 Product Rules

An important simplification occurs by using basis vectors adapted to the symmetry of the problem. Introducing the spherical basis vectors

$$\begin{aligned} \hat{\boldsymbol{r}} &= \sin\theta\cos\phi\,\hat{\boldsymbol{\imath}} + \sin\theta\sin\phi\,\hat{\boldsymbol{\jmath}} + \cos\theta\,\hat{\boldsymbol{k}} & (6) \\ \hat{\boldsymbol{\theta}} &= \cos\theta\cos\phi\,\hat{\boldsymbol{\imath}} + \cos\theta\sin\phi\,\hat{\boldsymbol{\jmath}} - \sin\theta\,\hat{\boldsymbol{k}} & (7) \\ \hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\boldsymbol{\imath}} + \cos\phi\,\hat{\boldsymbol{\jmath}} & (8) \end{aligned}$$

it is straightforward but messy to use (2) and (3) to calculate, for instance, that

$$\begin{aligned} \vec{\boldsymbol{\nabla}} \cdot \hat{\boldsymbol{r}} &= \frac{2}{r} & (9) \\ \vec{\boldsymbol{\nabla}} \times \hat{\boldsymbol{\phi}} &= \frac{\cot\theta}{r}\,\hat{\boldsymbol{r}} - \frac{1}{r}\,\hat{\boldsymbol{\theta}} & (10) \end{aligned}$$

either using the chain rule, or by rewriting (r, θ, ϕ) in terms of (x, y, z). If one now recalls that vector differentiation satisfies the product rules

$$\begin{aligned} \vec{\boldsymbol{\nabla}} \cdot (f\vec{\boldsymbol{F}}) &= \vec{\boldsymbol{\nabla}} f \cdot \vec{\boldsymbol{F}} + f\,(\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{F}}) & (11) \\ \vec{\boldsymbol{\nabla}} \times (f\vec{\boldsymbol{F}}) &= \vec{\boldsymbol{\nabla}} f \times \vec{\boldsymbol{F}} + f\,(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{F}}) & (12) \end{aligned}$$

then it is an easy matter to use (9) and (10) to show that

$$\begin{aligned} \vec{\boldsymbol{\nabla}} \cdot \left(\frac{1}{r^2}\,\hat{\boldsymbol{r}}\right) &= 0 & (13) \\ \vec{\boldsymbol{\nabla}} \times \left(\frac{1}{r\sin\theta}\,\hat{\boldsymbol{\phi}}\right) &= \vec{\boldsymbol{0}} & (14) \end{aligned}$$

for instance by using $r^2 = x^2 + y^2 + z^2$ and $r \sin\theta = \sqrt{x^2 + y^2}$. But these are precisely (4) and (5); the hard work here is in deriving the initial formulas (9) and (10). This approach is closely related to the concept of covariant differentiation in differential geometry.

2.3 Physics

After defining the divergence and curl in terms of a Cartesian basis, an introductory mathematics course typically goes on to prove the Divergence Theorem and Stokes' Theorem. If there is time — there often isn't — a geometric interpretation is then provided through the formulas

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{F}} = \lim_{S\to 0} \frac{1}{\text{Volume}(S)} \iint_S \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{S}} \tag{15}$$

$$(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{F}}) \cdot \hat{\boldsymbol{n}} = \lim_{C\to 0} \frac{1}{\text{Area}(C)} \oint_C \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}} \tag{16}$$

which relate divergence and curl to flux and circulation, respectively.

Physicists often turn this around, and use these formulas to *define* the divergence and curl, thus turning the Divergence Theorem and Stokes' Theorem into tautologies. These formulas are then used to *compute* the formulas for the divergence and curl in various coordinate systems. In spherical coordinates, for instance, this leads to formulas such as

$$\vec{\boldsymbol{\nabla}} \cdot (F^r \hat{\boldsymbol{r}}) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F^r\right) \tag{17}$$

$$\vec{\boldsymbol{\nabla}} \times (F^\phi \hat{\boldsymbol{\phi}}) = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta F^\phi\right)\hat{\boldsymbol{r}} - \frac{1}{r}\frac{\partial}{\partial r}\left(rF^\phi\right)\hat{\boldsymbol{\theta}} \tag{18}$$

from which (13) and (14) follow immediately. For a good, informal description of this approach, see [2].

2.4 Orthogonal Coordinates

The preceding approach generalizes naturally to any orthogonal coordinate system, that is, one in which the 3 coordinate directions are everywhere orthogonal. Typical examples are rectangular, cylindrical, and spherical coordinates, but there are many more.

A general orthogonal coordinate system (u, v, w) will have a line element of the form

$$ds^2 = f^2 du^2 + g^2 dv^2 + h^2 dw^2 \tag{19}$$

If we denote the unit vector fields in the coordinate directions by $\{\hat{\boldsymbol{u}}, \hat{\boldsymbol{v}}, \hat{\boldsymbol{w}}\}$, then we can expand any vector field $\vec{\boldsymbol{F}}$ as

$$\vec{\boldsymbol{F}} = F^u\,\hat{\boldsymbol{u}} + F^v\,\hat{\boldsymbol{v}} + F^w\,\hat{\boldsymbol{w}} \tag{20}$$

It is then a fairly simple computation [3] to derive the general formulas

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{F}} = \frac{1}{fgh} \frac{\partial f}{\partial u} (ghF^u) + ... \tag{21}$$

$$\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{F}} = \frac{1}{fg} \left[\frac{\partial}{\partial u} (gF^v) - \frac{\partial}{\partial v} (fF^u) \right] \hat{\boldsymbol{w}} + ... \tag{22}$$

using the formulas (15) and (16).

These formulas can hardly be called obvious. The corresponding formula for the gradient is much more natural. Starting from the chain rule, in the form

$$dk = \frac{\partial k}{\partial u} du + \frac{\partial k}{\partial v} dv + \frac{\partial k}{\partial w} dw \tag{23}$$

the all-important directional derivative, in the form

$$dk = \vec{\boldsymbol{\nabla}} k \cdot d\vec{\boldsymbol{r}} \tag{24}$$

together with the "square root" of the line element (in the sense $d\vec{\boldsymbol{r}} \cdot d\vec{\boldsymbol{r}} = ds^2$), given by

$$d\vec{\boldsymbol{r}} = f\, du\, \hat{\boldsymbol{u}} + g\, dv\, \hat{\boldsymbol{v}} + h\, dw\, \hat{\boldsymbol{w}} \tag{25}$$

we obtain

$$\vec{\boldsymbol{\nabla}} k = \frac{1}{f} \frac{\partial k}{\partial u} \hat{\boldsymbol{u}} + \frac{1}{g} \frac{\partial k}{\partial v} \hat{\boldsymbol{v}} + \frac{1}{h} \frac{\partial k}{\partial w} \hat{\boldsymbol{w}} \tag{26}$$

Examining (21) and (22), we see that there are special vector fields which are divergence or curl free, since

$$\vec{\boldsymbol{\nabla}} \cdot \frac{\hat{\boldsymbol{u}}}{gh} = 0 \tag{27}$$

$$\vec{\boldsymbol{\nabla}} \times \frac{\hat{\boldsymbol{u}}}{f} = \vec{\boldsymbol{0}} \tag{28}$$

These formulas can also be derived from the identities

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{\boldsymbol{F}}) = 0 \tag{29}$$

$$\vec{\nabla} \times \vec{\nabla} f = \vec{\boldsymbol{0}} \tag{30}$$

when one realizes that in orthogonal coordinates one has

$$\frac{\hat{\boldsymbol{u}}}{gh} = \vec{\boldsymbol{\nabla}} \times (v \vec{\boldsymbol{\nabla}} w) \tag{31}$$

$$\frac{\hat{\boldsymbol{u}}}{f} = \vec{\boldsymbol{\nabla}} u \tag{32}$$

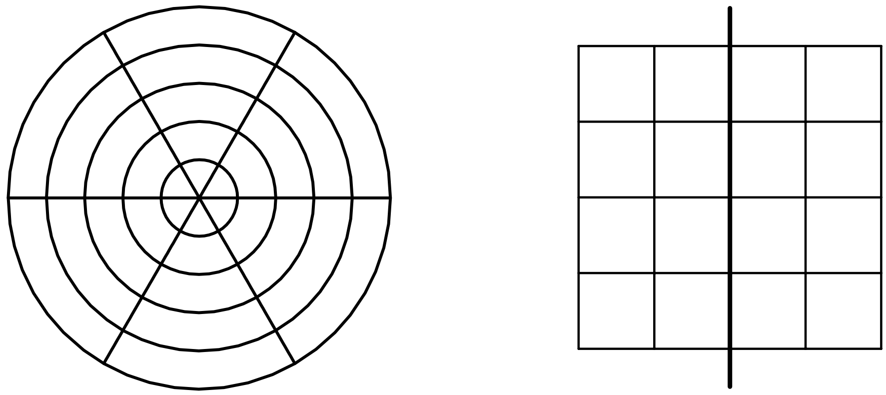

Figure 1: The first figure shows a horizontal slice of cylindrical coordinates, resulting in the usual polar coordinate grid. The second figure shows a vertical slice of cylindrical coordinates, through the z-axis (shown as a heavy line).

3 Electromagnetic Conic Sections

3.1 More Product Rules

As discussed in [4], the existence of a natural divergence-free basis along the lines of (27) can be used to reduce the computation of the divergence to the much simpler computation of the gradient. Similarly, the existence of a natural curl-free basis along the lines of (28) can be used to simplify the computation of the curl. In each case, this is accomplished using the appropriate product rule, (11) or (12), respectively. However, it is noteworthy that these two natural bases only agree in rectangular coordinates.

What if one could find a basis which was *both* divergence and curl free? In that case, one would never need to remember the formulas for the divergence and curl; all computations would reduce to the much simpler formula for the gradient.

Such a basis would also be of physical interest. A vector field which is both divergence and curl free solves Maxwell's vacuum equations, and can hence be interpreted as an electric or magnetic field. We are thus led to ask whether we can find a basis of electromagnetic fields.

We begin by considering several examples.

3.2 Plane

First of all, the rectangular basis $\{\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}\}$ is constant, and therefore, of course, both divergence and curl free. Each basis vector field must therefore describe an electromagnetic field. Which one? Consider an infinite parallel-plate capacitor, [1] with infinite separation between the plates. If the plates have equal but opposite (uniform) charge densities, then there is a constant electric field orthogonal to the plates. If, instead, the plates have equal but opposite (uniform) current densities, then there is a constant magnetic field parallel to the plates (but orthogonal to the currents).

[1] One plate is in fact sufficient. The advantage of two plates is that the field vanishes outside the capacitor.

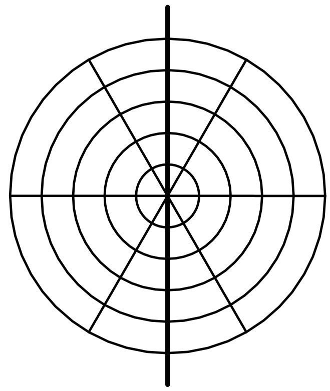

Figure 2: A vertical slice of spherical coordinates, showing the $r\theta$ coordinate grid.

3.3 Cylinder

Consider now the cylindrical coordinate system, defined by [2]

$$\begin{aligned} \sqrt{x^2+y^2} &= r \qquad (33) \\ \tan^{-1}\left(\frac{y}{x}\right) &= \phi \qquad (34) \end{aligned}$$

Horizontal (z = constant) and vertical ($\tan\phi$ = constant) slices through this coordinate system are shown in Figure 1. Denoting the adapted orthonormal basis as usual by $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{z}}\}$, we of course have $\hat{\boldsymbol{z}} \equiv \hat{\boldsymbol{k}}$, so that this basis vector field is both divergence and curl free. But what about the others?

The simplest cylindrical electromagnetic fields correspond to an infinite straight wire carrying either a uniform charge density or a uniform current density. It is straightforward to work out the corresponding fields: Up to scale, the electric field of the (positively) charged z-axis is

$$\vec{\boldsymbol{R}} = \frac{1}{r}\,\hat{\boldsymbol{r}} \qquad (35)$$

and the magnetic field of the (upward) current-carrying z-axis is

$$\vec{\boldsymbol{\Phi}} = \frac{1}{r}\,\hat{\boldsymbol{\phi}} \qquad (36)$$

Thus, an "electromagnetic" basis in this case is given by $\{\vec{\boldsymbol{R}}, \vec{\boldsymbol{\Phi}}, \hat{\boldsymbol{z}}\}$.

All of our remaining examples will be axially symmetric, and will thus have ϕ as a coordinate, $\hat{\boldsymbol{\phi}}$ as a basis vector field, and $\vec{\boldsymbol{\Phi}}$ as an electromagnetic basis vector field (although r will need to be expressed in terms of the given coordinates). We will omit further discussion of this case in (most of) the subsequent examples, and we will have no further use for horizontal slices analogous to the first figure in Figure 1.

[2]We use ϕ rather than θ for compatibility with our later examples.

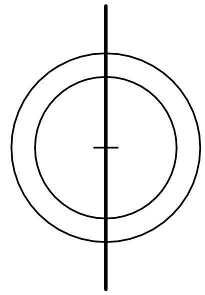

Figure 3: A spherical electric field. If the positive z-axis is given a uniform positive charge density, and the negative z-axis is given an equal and opposite charge density, the resulting field lines are spherical, that is, in the $\hat{\boldsymbol{\theta}}$ direction.

3.4 Sphere

What about the other standard coordinate system, namely spherical coordinates, defined implicitly by [3]

$$\sqrt{x^2+y^2} = r\sin\theta \tag{37}$$
$$z = r\cos\theta \tag{38}$$

(with ϕ as before), and shown in Figure 2? The adapted orthonormal basis is $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$, and we already know that

$$\vec{\boldsymbol{\Phi}} \equiv \frac{1}{r\sin\theta}\,\hat{\boldsymbol{\phi}} \tag{39}$$

is both divergence and curl free.

The only obvious spherical electromagnetic field is the electric field of a point charge, which is, up to scale

$$\vec{\boldsymbol{R}} = \frac{1}{r^2}\,\hat{\boldsymbol{r}} \tag{40}$$

This solves part of the problem. But what electromagnetic field, if any, looks like $\hat{\boldsymbol{\theta}}$? Somewhat surprisingly, it turns out there is one, namely the electric field of two half-infinite uniform line charges, with equal but opposite charge densities, as shown in Figure 3. Up to scale, the resulting divergence-free and curl-free basis vector field is

$$\vec{\boldsymbol{\Theta}} = \frac{1}{r}\,\hat{\boldsymbol{\theta}} \tag{41}$$

and an electromagnetic basis is given by $\{\vec{\boldsymbol{R}}, \vec{\boldsymbol{\Theta}}, \vec{\boldsymbol{\Phi}}\}$.

3.5 Ellipsoid & Hyperboloid

What about other, less common, orthogonal coordinate systems? Consider first ellipsoidal coordinates, defined by

$$\sqrt{x^2+y^2} = \sinh u \sin v \tag{42}$$
$$z = \cosh u \cos v \tag{43}$$

[3]Don't confuse the spherical radial coordinate with the cylindrical radial coordinate!

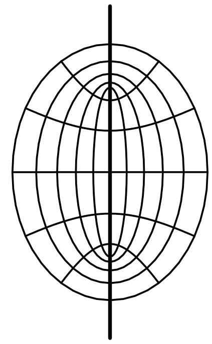

Figure 4: A vertical slice of ellipsoidal coordinates. The curves orthogonal to the ellipses are hyperbolas.

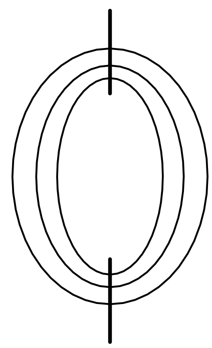

Figure 5: An ellipsoidal electric field. If the oppositely charged half-lines in the spherical example are separated by a finite gap, the resulting field lines are ellipsoidal.

as shown in Figure 4. The relevant orthonormal basis vectors are $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{v}}$; our goal is to find multiples of these which are both divergence and curl free, if possible.

With the wisdom of hindsight, that is, after having first computed the answer by brute force, it is clear that such vector fields do indeed exist. Consider the spherical model above, in which a multiple of $\hat{\boldsymbol{\theta}}$ was produced by two half-infinite line charges which were joined at the origin. Separate the two instead by a finite distance, as shown in Figure 5. The resulting electric field is just (proportional to)

$$\vec{\boldsymbol{V}} = \frac{1}{\sin v}\,\hat{\boldsymbol{v}} \tag{44}$$

and is therefore ellipsoidal. Similarly, the electric field of the "missing" finite line segment is just (proportional to)

$$\vec{\boldsymbol{U}} = \frac{1}{\sinh u}\,\hat{\boldsymbol{u}} \tag{45}$$

which is hyperboloidal, as shown in Figure 6 [4] An electromagnetic basis in this case is therefore given by $\{\vec{\boldsymbol{U}}, \vec{\boldsymbol{V}}, \vec{\boldsymbol{\Phi}}\}$.

[4]It is instructive to consider this latter example as a "stretched out" point charge.

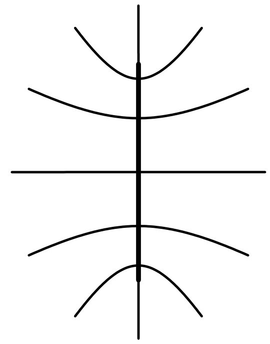

Figure 6: A hyperboloidal electric field, the electric field of a uniformly charged line segment.

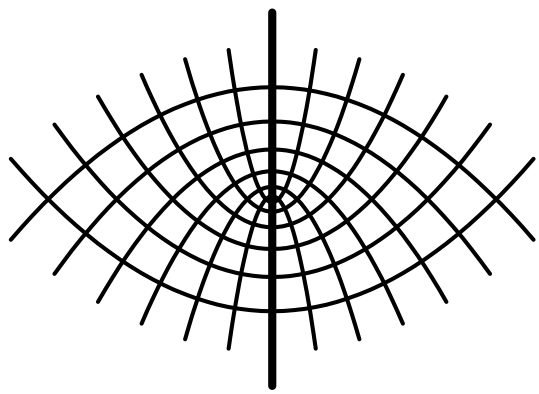

Figure 7: A vertical slice of paraboloidal coordinates. Both families of orthogonal curves are parabolas.

3.6 Paraboloid

Moving right along, now consider paraboloidal coordinates, defined by

$$\sqrt{x^2+y^2} = uv \tag{46}$$

$$z = \frac{1}{2}(u^2-v^2) \tag{47}$$

and shown in Figure 7. Do there exist multiples of $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{v}}$ which are both divergence and curl free?

Again, with the wisdom of hindsight the answer is clearly yes. The electric field of a half-infinite, uniform line charge is shown in Figure 8, corresponding to

$$\vec{\boldsymbol{U}} = \frac{1}{u\sqrt{u^2+v^2}}\,\hat{\boldsymbol{u}} \tag{48}$$

$$\vec{\boldsymbol{V}} = \frac{1}{v\sqrt{u^2+v^2}}\,\hat{\boldsymbol{u}} \tag{49}$$

respectively.

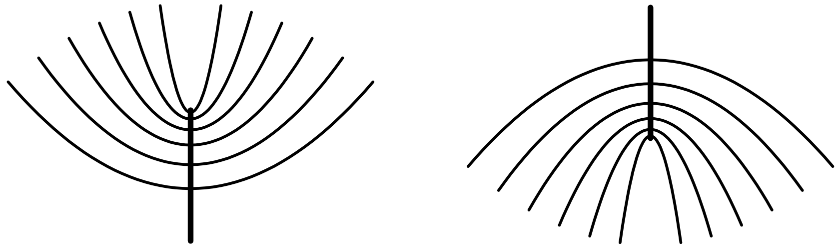

Figure 8: Two paraboloidal electrical fields, in each case the electric field of a half-infinite uniform line charge.

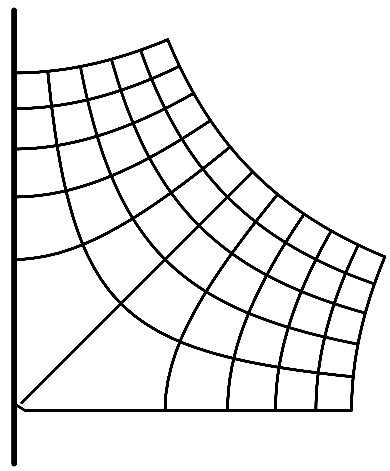

Figure 9: A vertical slice of hyperboloidal coordinates. Both families of orthogonal curves are hyperbolas.

3.7 Another Hyperboloid

Buoyed by our success, let us finally consider hyperboloidal coordinates, defined by

$$\sqrt{x^2+y^2} = \sqrt{\sqrt{u^2+v^2}-u} \tag{50}$$

$$z = \sqrt{\sqrt{u^2+v^2}+u} \tag{51}$$

and shown in Figure 9. We have

$$\hat{\boldsymbol{u}} = \frac{-x\hat{\boldsymbol{\imath}} - y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}}{\sqrt{x^2+y^2+z^2}} \tag{52}$$

$$\hat{\boldsymbol{v}} = \frac{zx\hat{\boldsymbol{\imath}} + zy\hat{\boldsymbol{\jmath}} + (x^2+y^2)\hat{\boldsymbol{k}}}{\sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2}} \tag{53}$$

and we seek multiples of $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{v}}$ which are both divergence and curl free.

There aren't any.

3.8 General Case

So when does it work?

Given a vector field $\vec{\boldsymbol{G}}$, we ask whether there exists a function λ such that $\lambda\vec{\boldsymbol{G}}$ is both divergence and curl free, that is, such that

$$\vec{\boldsymbol{\nabla}} \cdot \lambda\vec{\boldsymbol{G}} = 0 \tag{54}$$

$$\vec{\boldsymbol{\nabla}} \times \lambda\vec{\boldsymbol{G}} = \vec{\boldsymbol{0}} \tag{55}$$

Using the product rules (11) and (12), we can rewrite these conditions as

$$\vec{\boldsymbol{\nabla}}\lambda \cdot \vec{\boldsymbol{G}} = -\lambda\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{G}} \tag{56}$$

$$\vec{\boldsymbol{\nabla}}\lambda \times \vec{\boldsymbol{G}} = -\lambda\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{G}} \tag{57}$$

On the other hand, the identity

$$\vec{\boldsymbol{u}} \times (\vec{\boldsymbol{v}} \times \vec{\boldsymbol{w}}) = (\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{w}})\,\vec{\boldsymbol{v}} - (\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{v}})\,\vec{\boldsymbol{w}} \tag{58}$$

leads to

$$(\vec{\boldsymbol{\nabla}}\lambda \times \vec{\boldsymbol{G}}) \times \vec{\boldsymbol{G}} = (\vec{\boldsymbol{\nabla}}\lambda \cdot \vec{\boldsymbol{G}})\,\vec{\boldsymbol{G}} - |\vec{\boldsymbol{G}}|^2\,\vec{\boldsymbol{\nabla}}\lambda \tag{59}$$

Rearranging terms and using (56) and (57) then yields

$$\vec{\boldsymbol{\nabla}}\lambda = (\vec{\boldsymbol{\nabla}}\lambda \cdot \vec{\boldsymbol{G}})\,\frac{\vec{\boldsymbol{G}}}{|\vec{\boldsymbol{G}}|^2} - (\vec{\boldsymbol{\nabla}}\lambda \times \vec{\boldsymbol{G}}) \times \frac{\vec{\boldsymbol{G}}}{|\vec{\boldsymbol{G}}|^2} \tag{60}$$

$$= \lambda\left((\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{G}}) \times \frac{\vec{\boldsymbol{G}}}{|\vec{\boldsymbol{G}}|^2} - (\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{G}})\frac{\vec{\boldsymbol{G}}}{|\vec{\boldsymbol{G}}|^2}\right) \tag{61}$$

Putting this all together, the necessary and sufficient condition that a suitable λ exist is that

$$\vec{\boldsymbol{\nabla}} \times \left((\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{G}}) \times \frac{\vec{\boldsymbol{G}}}{|\vec{\boldsymbol{G}}|^2} - (\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{G}})\frac{\vec{\boldsymbol{G}}}{|\vec{\boldsymbol{G}}|^2}\right) = 0 \tag{62}$$

Both directions follow immediately from the fact that the term in parentheses is just $\vec{\boldsymbol{\nabla}} \ln \lambda$. Thus, if λ exists, then (62) is satisfied due to the identity (30), whereas if (62) is satisfied, then there exists a (local) potential function, which is $\ln \lambda$.

4 Discussion

We have demonstrated a possible alternative way to compute the divergence and curl in certain standard cases. For instance, in spherical coordinates, one really need only remember that $\{\vec{\boldsymbol{R}}, \vec{\boldsymbol{\Theta}}, \vec{\boldsymbol{\Phi}}\}$ is an electromagnetic basis — ideally by recalling the corresponding electromagnetic fields. The divergence and curl are then easily computed from formulas like

$$\vec{\boldsymbol{\nabla}} \cdot (f\vec{\boldsymbol{R}}) = \vec{\boldsymbol{\nabla}} f \cdot \vec{\boldsymbol{R}} \tag{63}$$

$$\vec{\boldsymbol{\nabla}} \times (f\vec{\boldsymbol{R}}) = \vec{\boldsymbol{\nabla}} f \times \vec{\boldsymbol{R}} \tag{64}$$

Yes, this requires knowing how to compute the gradient in spherical coordinates, but this can easily be rederived as needed from the geometrically obvious formula

$$d\vec{\boldsymbol{r}} = dr\,\hat{\boldsymbol{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}} \tag{65}$$

Turning to the general case, the condition (62) not only characterizes the vector fields $\vec{\boldsymbol{G}}$ which can be rescaled so as to be both divergence and curl free, it also provides an explicit algorithm for determining λ. There is another, simpler characterization, but without this property.

Requiring $\vec{\boldsymbol{F}}$ to be curl free means that (locally)

$$\vec{\boldsymbol{F}} = \vec{\boldsymbol{\nabla}} f \tag{66}$$

In particular, since we are assuming $\vec{\boldsymbol{F}} = \lambda\vec{\boldsymbol{G}}$, this forces the original vector field $\vec{\boldsymbol{G}}$ to be orthogonal to the surfaces $\{\lambda = \text{constant}\}$. Thus, a necessary condition on $\vec{\boldsymbol{G}}$ is that it be *hypersurface orthogonal*. This condition is always satisfied for the examples considered here, constructed from a coordinate system.

The condition that $\vec{\boldsymbol{F}}$ be divergence free imposes the further condition that

$$\triangle f = 0 \tag{67}$$

so that $\vec{\boldsymbol{F}}$ must be the gradient of a *harmonic* function. Thus, the question of which coordinate systems admit basis vectors which can (all) be rescaled so as to be divergence and curl free is equivalent to the question of which coordinate systems can themselves be rescaled so as to be harmonic coordinates.

We conclude by noting that harmonic functions in 2 dimensions are closely related to analytic functions. A vector field $\vec{\boldsymbol{F}} = P\hat{\boldsymbol{\imath}} + Q\hat{\boldsymbol{\jmath}}$ is divergence and curl free if and only if $P - iQ$ is analytic, since

$$\frac{\partial}{\partial\bar{z}}(P - iQ) = \frac{1}{2}\left(\vec{\boldsymbol{\nabla}}\cdot\vec{\boldsymbol{F}} - i\,|\vec{\boldsymbol{\nabla}}\times\vec{\boldsymbol{F}}|\right) \tag{68}$$

A similar statement can be made in 3 dimensions, using quaternions in place of the complex numbers.

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