THE VECTOR CALCULUS BRIDGE PROJECT

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I: Mathematics ≠ Physics
II: The Bridge Project
III: A Radical View of Calculus
Support

- Oregon State University
  - Department of Mathematics
  - Department of Physics
- Grinnell College
  - Noyce Visiting Professorship
- Mount Holyoke College
  - Hutchcroft Fund
- National Science Foundation
  - DUE-9653250
  - DUE-0088901
- Oregon Collaborative for Excellence in the Preparation of Teachers
How many college students took mathematics courses in 2000?

- precalculus: 2,000,000
- calculus: 700,000
- statistics: 250,000
- everything else: 250,000

MAA Committee on the Undergraduate Program in Mathematics
CUPM Curriculum Guide
http://www.maa.org/cupm
I: Suppose $T(x, y) = k(x^2 + y^2)$ where $k$ is a constant. What is $T(r, \theta)$?

II: Suppose the temperature on a rectangular slab of metal is given by $T(x, y) = k(x^2 + y^2)$ where $k$ is a constant. What is $T(r, \theta)$?

**Choices**

A: $T(r, \theta) = kr^2$

B: $T(r, \theta) = k(r^2 + \theta^2)$
MATH

\[ T = f(x, y) = k(x^2 + y^2) \]
\[ T = g(r, \theta) = kr^2 \]

PHYSICS

\[ T = T(x, y) = k(x^2 + y^2) \]
\[ T = T(r, \theta) = kr^2 \]

Differential Geometry!

\[ T(x, y) \leftrightarrow T \circ (x, y)^{-1} \]
\[ T(r, \theta) \leftrightarrow T \circ (r, \theta)^{-1} \]
Find the directional derivative of the function at the given point in the direction of $\vec{v}$.

(a) $f(x, y) = x/y$, $(6, -2)$, $\vec{v} = (-1, 3)$

(b) $g(s, t) = s^2 e^t$, $(2, 0)$, $\vec{v} = \hat{i} + \hat{j}$

(c) $g(r, \theta) = e^{-r} \sin \theta$, $(0, \pi/3)$, $\vec{v} = 3 \hat{i} - 2 \hat{j}$
I: Physics is about things.

(a) What sort of a beast is it?
- Scalar fields
- Units
- Coordinates
- Time

(b) Physics is independent of coordinates.
- Vectors as arrows
- Geometry of dot and cross products

(c) Graphs are about the relationships of physical things.
- Fundamental physics is three dimensional.
- 3d graphs of functions of 2 vars are misleading
- Hills are not good examples of functions of two variables
- Use of color
- Graphs of waves are misinterpreted

(d) Fundamental physics is highly symmetric.
- Spheres and cylinders vs. paraboloids
- Interesting physics problems can involve trivial math
- Use of $\hat{r}$, $\hat{\theta}$, $\hat{\phi}$
II: Physicists can’t change the problem.

(a) Physics involves the creative synthesis of multiple ideas.
(b) Physics problems may not be well-defined math problems.
   • No preferred coordinates or independent variables.
   • No parameterization.
   • Unknowns don’t have names.
   • Getting to a well-defined math problem is part of the problem
   • If you can’t add units, it’s a poor physics problem.
(c) Physics problems don’t fit templates.
   • Template problem-solving vs. skills
   • A few key ideas are remembered best later
(d) Physics involves the interplay of multiple representations.
   • Dot product
THE BRIDGE PROJECT

- Small group activities
- Instructor’s guide (in preparation)
- CWU, MHC, OSU, UPS, UWEC
- [http://www.physics.orst.edu/bridge](http://www.physics.orst.edu/bridge)
- Workshops, MAA Minicourse

- **Differentials** (*Use what you know!*)
- **Multiple representations**
- **Symmetry** (*adapted bases, coordinates*)
- **Geometry** (*vectors, div, grad, curl*)
PEOPLE

- Barbara Edwards
- Tom Dick, John Lee
- Johnner Barrett, Sam Cook
- Stuart Boersma, Marc Goulet, Martin Jackson
- David Griffiths, Harriet Pollatsek
- Bill McCallum, Jim Stewart
PUBLICATIONS


I: In the small town of Coriander, the library can be found by starting at the center of the town square, walking 25 meters north, turning 90° to the right, and walking a further 60 meters.

II: It turns out that magnetic north in Coriander is approximately 14° degrees east of true north. If you use a compass to find the library (!), the above directions will fail. Instead, you must walk 39 meters in the direction of magnetic north, turn 90° to the right, and walk a further 52 meters.

Where on Earth is Coriander?
VECTOR DIFFERENTIALS

\[ d\mathbf{r} = dx \, \hat{i} + dy \, \hat{j} \]

\[ d\mathbf{r} = dr \, \hat{r} + r \, d\theta \, \hat{\theta} \]

\[ ds = |d\mathbf{r}| \]
\[ d\mathbf{S} = d\mathbf{r}_1 \times d\mathbf{r}_2 \]
\[ dS = |d\mathbf{r}_1 \times d\mathbf{r}_2| \]
\[ dV = (d\mathbf{r}_1 \times d\mathbf{r}_2) \cdot d\mathbf{r}_3 \]
\[ df = \nabla f \cdot d\mathbf{r} \]
MATHEMATICIANS’ LINE INTEGRALS

• Start with Theory

\[ \int \vec{F} \cdot d\vec{r} = \int \vec{F} \cdot \hat{T} \, ds \]

\[ = \int \vec{F}(\vec{r}(t)) \cdot \frac{\vec{r}'(t)}{|\vec{r}'(t)|} |\vec{r}'(t)| \, dt \]

\[ = \int \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt \]

\[ = \ldots = \int P \, dx + Q \, dy + R \, dz \]

• Do examples starting from next-to-last line

Need parameterization \( \vec{r} = \vec{r}(t) \)
PHYSICISTS’ LINE INTEGRALS

• Theory
  – Chop up curve into little pieces $d\vec{r}$.
  – Add up components of $\vec{F}$ parallel to curve
    (times length of $d\vec{r}$)

• Do examples directly from $\vec{F} \cdot d\vec{r}$

Need $d\vec{r}$ along curve
\[ \vec{F}(x, y) = \frac{-y \hat{i} + x \hat{j}}{x^2 + y^2} \quad \vec{r} = x \hat{i} + y \hat{j} \]

\[ x = 2 \cos \theta \]
\[ y = 2 \sin \theta \]

\[
\int \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} \vec{F}(x(\theta), y(\theta)) \cdot \vec{r}'(x(\theta), y(\theta)) \, d\theta \\
= \int_0^{\pi/2} \frac{1}{2} (-\sin \theta \hat{i} + \cos \theta \hat{j}) \cdot 2(-\sin \theta \hat{i} + \cos \theta \hat{j}) \, d\theta \\
= \ldots = \frac{\pi}{2}
\]
\[ \vec{F} = \hat{\theta} \frac{\theta}{r} \]

\[ d\vec{r} = r \, d\theta \, \hat{\theta} \]

**I:** \( |\vec{F}| = \text{const}; \vec{F} \parallel d\vec{r} \implies \)

\[ \int \vec{F} \cdot d\vec{r} = \frac{1}{2}(2 \frac{\pi}{2}) \]

**II:** Do the dot product \( \leftarrow \)

\[ \int \vec{F} \cdot d\vec{r} = \int_0^{\frac{\pi}{2}} \frac{\theta}{2} \cdot 2 \, d\theta \, \hat{\theta} = \int_0^{\frac{\pi}{2}} d\theta = \frac{\pi}{2} \]
\[ \mathbf{F}(x, y) = \frac{-y \mathbf{i} + x \mathbf{j}}{x^2 + y^2} \quad \mathbf{r} = x \mathbf{i} + y \mathbf{j} \]

\[
\begin{align*}
y &= 1 - x^2 \\
x &= 1 - t \\
y &= 1 - (1 - t^2)^2 = 2t - t^2
\end{align*}
\]

\[
\int \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(x(t), y(t)) \cdot \mathbf{r}'(x(t), y(t)) \, dt
\]

\[
= \ldots = \int_0^1 \frac{2 - 2t + t^2}{1 - 2t + 5t^2 - 4t^3 + t^4} \, dt = \frac{\pi}{2}
\]
\[ \mathbf{d\vec{r}} = (\hat{i} - 2x \hat{j}) \, dx \]

\[
\int \mathbf{\bar{F}} \cdot d\mathbf{\vec{r}} = \int_{0}^{1} \frac{-(1 - x^2) \hat{i} + x \hat{j}}{x^2 + (1 - x^2)^2} \cdot (\hat{i} - 2x \hat{j}) \, dx
\]

\[
= - \int_{0}^{1} \frac{1 + x^2}{1 - x^2 + x^4} \, dx
\]

Oops! — answer should be positive

\[
\int \mathbf{\bar{F}} \cdot d\mathbf{\vec{r}} = \int_{0}^{1} \frac{1 + x^2}{1 - x^2 + x^4} \, dx = \frac{\pi}{2}
\]
\[ \vec{F}(x, y) = \frac{-y \hat{i} + x \hat{j}}{x^2 + y^2} \quad d\vec{r} = dx \hat{i} + dy \hat{j} \]

\[ y = 1 - x^2 \]
\[ dy = -2x \, dx \]

\[ \int \vec{F} \cdot d\vec{r} = \int_1^0 \frac{-(1 - x^2) \hat{i} + x \hat{j}}{x^2 + (1 - x^2)^2} \cdot (dx \hat{i} - 2x \, dx \hat{j}) \]
\[ = -\int_1^0 \frac{1 + x^2}{1 - x^2 + x^4} \, dx \]
\[ = \frac{\pi}{2} \]
The central idea in calculus is not the limit.
The central idea of derivatives is not slope.
The central idea of integrals is not area.
The central idea of curves and surfaces is not parameterization.
The central representation of a function is not its graph.
A RADICAL VIEW OF CALCULUS

- The central idea in calculus is the differential.
- The central idea of derivatives is rate of change.
- The central idea of integrals is total amount.
- The central idea of curves and surfaces is “use what you know”.
- The central representation of a function is data attached to the domain.