Part I: Complex Analysis and Linear Algebra

- Do any two problems in Part CA and any two problems in Part LA.
- Your solutions should include all essential mathematical details; please write them up as clearly as possible.
- State explicitly any standard theorems that are needed to justify your reasoning.
- You have three hours to complete Part I of the exam.
- In problems with multiple parts, the individual parts may be weighted differently in grading.

Part CA

1. Find the Laurent expansion $\sum_{n=-\infty}^{\infty} a_n z^n$ of the function
   \[
   f(z) = \frac{1}{(z - 1)(z - 2)}
   \]
   (a) in the region $1 < |z| < 2$;
   (b) in the region $|z| > 2$.

2. Use the Cauchy Integral Formula to prove that if a function $f(z)$ is analytic in a domain $D$ and if $z_0$ is a point of $D$, then $f(z)$ has a power series expansion in some open disk centered at $z_0$. (Do not appeal directly to Taylor’s theorem or Laurent’s theorem.) State explicitly any other standard theorems that are needed to justify your solution.

3. Let $f(z)$ be an analytic function in an open region of $\mathbb{C}$ containing the closed unit disc $D$. Suppose $f(0) = 1$ and $|f(z)| > 1$ whenever $|z| = 1$. Show that $f(z)$ has a zero in $D$. 
Part LA

1. Suppose $A = \begin{bmatrix} 0 & 2 \\ -3 & 5 \end{bmatrix}$. Find $A^{10,000}$. Justify your answer.

2. Let $V$ be an $n$-dimensional complex inner product space, $T$ be a linear operator on $V$, $W$ be a $T$-invariant subspace of $V$ with $\dim W = m$, and $T^*$ be the adjoint of $T$.

   (a) Give an example to show that $W$ need not be $T^*$-invariant. Verify that your example is $T$-invariant but not $T^*$-invariant.

   (b) Assume that $W$ is both $T$ and $T^*$-invariant. Show there exists a basis $\beta$ for $V$ such that the matrices of $T$ and $T^*$ with respect to $\beta$, $[T]_{\beta}$ and $[T^*]_{\beta}$, have the forms

   $[T]_{\beta} = \begin{bmatrix} A & O \\ O & B \end{bmatrix}$ and $[T^*]_{\beta} = \begin{bmatrix} C & O \\ O & D \end{bmatrix}$

   where $A$ and $C$ are $m \times m$ matrices and $B$ and $D$ are $(n-m) \times (n-m)$ matrices.

3. Suppose $n \times n$ complex matrices $A$ and $B$ have the same characteristic polynomial $p(x)$ and the same minimal polynomial $q(x)$. (Of course, $p(x)$ may not be equal to $q(x)$.) Can we conclude that $A$ and $B$ are similar (i.e., $A = CBC^{-1}$ for some invertible $n \times n$-matrix $C$),

   (a) if $n = 3$?

   (b) if $n = 4$?

   In each part give a proof or a counterexample.
Part II: Real Analysis

- Do any four of the problems in Part II.
- Your solutions should include all essential mathematical details; please write them up as clearly as possible.
- State explicitly any standard theorems that are needed to justify your reasoning.
- You have three hours to complete Part II of the exam.
- In problems with multiple parts, the individual parts may be weighted differently in grading.

1. Let \( f : X \rightarrow X \) be a map from a metric space into itself. A point \( z \in X \) is a fixed point of \( f \) if \( f(z) = z \). Let \( \varepsilon > 0 \). A point \( w \in X \) is an \( \varepsilon \)-fixed point of \( f \) if \( d(f(w), w) < \varepsilon \).

   (a) Prove: If \( X \) is a compact metric space, \( f : X \rightarrow X \) is a continuous function, and if for every \( \varepsilon > 0 \) \( f \) has an \( \varepsilon \)-fixed point, then \( f \) has a fixed point.

   (b) Prove the following statement or give a counter example: If \( X \) is a metric space, \( f : X \rightarrow X \) is a continuous function, and if for every \( \varepsilon > 0 \), \( f \) has an \( \varepsilon \)-fixed point, then \( f \) has a fixed point.

2. Let \( f \in L^p(\mathbb{R}) \) for some \( 1 \leq p < \infty \).

   (a) Show that
   \[
   \lim_{x \to \infty} \int_x^{x+1} f(t) \, dt = 0.
   \]

   (b) Show, by way of example, that the assertion of part (a) may fail if \( p = \infty \).
3. Two norms $\| \cdot \|_\alpha$ and $\| \cdot \|_\beta$ on a vector space $V$ are equivalent if there are positive constants $m$ and $M$ such that

$$m \| x \|_\alpha \leq \| x \|_\beta \leq M \| x \|_\alpha$$

for all $x \in V$.

(a) Prove that any two norms on $\mathbb{R}^n$ are equivalent. \textit{Hint.} For any norm $\| \cdot \|$ on $\mathbb{R}^n$ consider the function $f(x) = \| x \|$ on the set

$$\left\{ x = (x_1, \ldots, x_n) : \sum_{i=1}^{n} |x_i| = 1 \right\}.$$

(b) Show that the following norms on $C[0, 1]$, the continuous real-valued functions on $[0, 1]$, are not equivalent:

$$\| f \| = \max_{[0,1]} |f(x)| \quad \text{and} \quad \| f \|_1 = \int_0^1 |f(x)| \, dx$$

4. Let $f$ be an $L^1$-function on $[0, \infty)$.

(a) Show that if $f$ is uniformly continuous on $[0, \infty)$ then

$$\lim_{t \to \infty} f(t) = 0.$$

(b) Show, by way of example, that the conclusion of part (a) may fail if $f$ is assumed to be continuous (and $L^1$) but not uniformly continuous on $[0, \infty)$. 

4
5. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a non-negative $L^{p_0}$ function, where $0 < p_0 < \infty$. Show that
\[
\lim_{p \to 0^+} \int_{\mathbb{R}} f^p \, d\nu = \nu \left( \{ x \in X : f(x) \neq 0 \} \right)
\]
where $\nu$ is the usual Lebesgue measure on the real line.

Hint: Write
\[
\int_{\mathbb{R}} f^p \, d\nu = \int_{X_0} f^p \, d\nu + \int_{X_1} f^p \, d\nu + \int_{X_2} f^p \, d\nu,
\]
where $X_0 = \{ x \in \mathbb{R} : f(x) = 0 \}$, $X_1 = \{ x \in \mathbb{R} : 0 < f(x) < 1 \}$, and $X_2 = \{ x \in \mathbb{R} : f(x) \geq 1 \}$.

6. Let $V$ be the inner product space of all continuous real-valued functions on $[-1, 1]$ with the inner product
\[
\langle f, g \rangle = \int_{-1}^{1} f(t)g(t) \, dt.
\]

Let $W$ be the subspace of $V$ consisting of odd functions, i.e., $h \in V$ lies in $W$ if and only if $h(-x) = -h(x)$. Find the orthogonal complement $W^\perp$ of $W$. Justify your answer.