Department of Mathematics
Qualifying Examination
Fall 2007

Part I: Real Analysis

• Do any four of the problems in Part I.

• Your solutions should include all essential mathematical details; please write them up as clearly as possible.

• State explicitly including all hypotheses any standard theorems that are needed to justify your reasoning.

• You have three hours to complete Part I of the exam.

• In problems with multiple parts, the individual parts may be weighted differently in grading.
1. Suppose $f \in L^{p_0}(\mathbb{R}^n) \cap L^{\infty}(\mathbb{R}^n)$ holds, for some $1 \leq p_0 < \infty$.
   (a) Prove that $f \in L^p(\mathbb{R}^n)$ for $p$ with $p_0 < p < \infty$.
   (b) Prove that $\lim_{p \to \infty} \|f\|_p = \|f\|_{\infty}$.
   (c) Give an example of a function in $L^{p_0}(\mathbb{R}^n)$, for some $1 \leq p_0 < \infty$, but which is not in $L^p(\mathbb{R}^n)$, for some $p$ with $p_0 < p < \infty$.

2. Let $m^*$ denote Lebesgue outer measure on $\mathbb{R}$. Suppose $f : \mathbb{R} \to \mathbb{R}$ satisfies
   $$ |f(x) - f(y)| \leq C |x - y| $$
   for all $x, y \in \mathbb{R}$ (here $0 < C < \infty$).
   (a) Prove that $m^*[f(A)] \leq C m^*(A)$ holds for every $A \subset \mathbb{R}$.
   (b) Prove that if $A \subset \mathbb{R}$ is a Lebesgue measurable set, then so is $f(A)$.

3. (a) Show that the metric space of continuous functions on the interval $[0, 1]$ equipped with the $L^2$-metric is incomplete.
   (b) By the diameter of a subset $A$ of a metric space $X$ is meant the number
   $$ d(A) = \sup_{x,y \in A} \rho(x,y). $$
   where $\rho$ denotes the metric. Suppose $X$ is complete, and let $\{A_n\}$ be a sequence of closed nonempty subsets of $X$ nested in the sense that
   $$ A_1 \supset A_2 \supset \ldots \supset A_n \supset \ldots. $$
   Suppose further that
   $$ \lim_{n \to \infty} d(A_n) = 0. $$
   Prove that the intersection $\bigcap_{n=1}^{\infty} A_n$ consists of a single point.

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4. Let $M$ be a bounded subset of $C([a, b])$. Consider the set $S \subset C([a, b])$ of all functions $F$ such that

$$F(x) = \int_{a}^{x} f(t) \, dt$$

for some $f$ in $M$. Show that the closure of $S$ is a compact subset of $C([a, b])$. When solving this problem, state precisely the hypothesis and conclusion of any major theorem that you are using.

5. (a) Let $f : [-1, 1] \to \mathbb{R}$ be continuous. Suppose $\int_{-1}^{1} f(x) x^n \, dx = 0$ holds for $n = 0, 1, 2, \ldots$. Prove that $f(x) = 0$ holds for all $x \in [-1, 1]$.

(b) Let $\varphi_n$, $n = 1, 2, \ldots$ be a sequence of functions in $L^2([0, 2\pi])$ such that $\int_{0}^{2\pi} \varphi_n(t) \varphi_m(t) \, dt$ is equal to 1 for $n = m$ and vanishes for $n \neq m$. If $A \subset [0, 2\pi]$ and $A$ is measurable, prove that

$$\lim_{n \to \infty} \int_{A} \varphi_n(x) \, dx = 0.$$

6. Let $\{ q_i \}_{i=0}^{\infty}$ be an enumeration of the rationals in the unit interval $[0, 1]$. Suppose that $q_0 = 0$ and $q_1 = 1$. Define a function $f$ on the rationals in the unit interval by setting $f(q_0) = 0$, setting $f(q_1) = 1$, and, for $n \geq 2$, recursively setting

$$f(q_n) = \frac{f(q_n^-) + f(q_n^+)}{2}$$

where

$$q_n^- = \max\{ q_i : i = 0, 1, \ldots, n-1 \text{ and } q_i < q_n \},$$

$$q_n^+ = \min\{ q_i : i = 0, 1, \ldots, n-1 \text{ and } q_n < q_i \}.$$

So for instance, $q_2^- = q_0$, $q_2^+ = q_1$, and $f(q_2) = [f(q_0) + f(q_1)]/2 = 1/2$.

(a) Prove that $f$ is monotone on the rationals in $[0, 1]$.

(b) Prove that $f$ is continuous on the rationals in $[0, 1]$.

(c) Can $f$ be extended continuously to all real numbers in $[0, 1]$, and why or why not?
Part II: Complex Analysis and Linear Algebra

- Do any two problems in Part CA and any two problems in Part LA.
- Your solutions should include all essential mathematical details; please write them up as clearly as possible.
- State explicitly including all hypotheses any standard theorems that are needed to justify your reasoning.
- You have three hours to complete Part II of the exam.
- In problems with multiple parts, the individual parts may be weighted differently in grading.
1. Suppose $f$ is an analytic function on the unit disc, $D \equiv \{ |z| \leq 1 \}$. Suppose $f(0) = 0$ and $|f(z)| \leq 1$, for all $z \in D$. Show that $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$, for all $z \in D$.

2. Let $P_n(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!}$. Show that for every given positive real number $r > 0$, there exists a positive integer $M$ such that for every $n \geq M$ all zeros of the polynomial $P_n(z)$ lie outside the circle $|z| = r$.

3. Suppose $f$ is a complex function defined on the open unit disc, $|z| < 1$.

   (a) Show or give a counterexample: If $f^2$ is analytic on $D$, then $f$ is analytic on $D$.

   (b) Show that if $f^2$ and $f^3$ are analytic on $D$, then $f$ is analytic on $D$.  

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Part: Linear Algebra

1. Let $F$ be a field. For $m$ and $n$ positive integers, let $M_{m,n}$ be the vector space of $m \times n$ matrices over $F$. Fix $m$ and $n$, and fix matrices $A$ and $B$ in $M_{m,n}$. Define the linear transformation $T$ from $M_{n,m}$ to $M_{m,n}$ by $T(X) = AXB$. Prove that if $m \neq n$, then $T$ is not invertible.

2. Let $S$ be the subspace of $M_{n,n}$ (the vector space of all real $n \times n$ matrices) generated by all matrices of the form $AB - BA$ with $A$ and $B$ in $M_{n,n}$. Prove that $\dim(S) = n^2 - 1$.

3. Let

$$ M = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. $$

(a) Find the minimal and characteristic polynomials of $M$.
(b) Is $M$ similar to a diagonal matrix $D$ over $\mathbb{R}$? If so, find such a $D$.
(c) Repeat part (b) with $\mathbb{R}$ replaced by $\mathbb{C}$ and also by the field $\mathbb{Z}/5\mathbb{Z}$. 

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