Optimal placement of Marine Protected Areas: a trade-off between fisheries goals and conservation efforts

Patrick De Leenheer

Abstract—Marine Protected Areas (MPAs) are regions in the ocean or along coastlines where fishing is controlled to avoid the reduction or elimination of fish populations. A central question is where exactly to establish an MPA. We cast this as an optimal problem along a one-dimensional coast-line, where fish are assumed to move diffusively, and are subject to recruitment, natural death and harvesting through fishing. The functional being maximized is a weighted sum of the average fish density and the average fishing yield. It is shown that optimal controls exist, and that the location of the MPA is determined by two key model parameters, namely the size of the coast, and the weight of the average fish density in the functional.

Index Terms—optimal control, fisheries.

I. INTRODUCTION

Marine Protected Areas (MPAs) [7] are regions in oceans or along coastlines where fishing is controlled. MPAs have been proposed as a fisheries management tool and contrast more traditional approaches which rely on limiting spatially uniform harvesting rates. The purpose of this paper is to present a mathematical framework to aid in the decision of whether or not it would be beneficial to introduce an MPA, and if so, where to implement it. A novel objective measure capturing the effect of the MPA is proposed. It takes the form of a weighted sum consisting of the yield, and the average fish density. This leads to a trade-off problem, and the natural context to consider it is provided by optimal control theory [4], [5]. This paper is not the first to propose the use of optimal control in the context of MPAs, and follows the lead of [6]. However, both the model, the analysis, and the results obtained here, deviate from those in [6] in several respects. Details of some proofs are omitted due to space constraints, but can be found in [2].

II. THE PROBLEM

Consider the following model:

$$
\begin{aligned}
U_T &= DU_{XX} + R - \mu U - H(X)U, \\
U(-L/2, T) &= U(L/2, T) = 0,
\end{aligned}
\tag{1}
$$

for all $T \geq 0$ and $X \in (-L/2, L/2)$. Here, points along the scalar coastline of length $L > 0$ are represented by the spatial variable X taking values in the interval $[-L/2, L/2]$, and $U(X, T)$ denotes the fish density at location X and time

P. De Leenheer is with the Department of Mathematics and the Department of Zoology of the Oregon State University, Corvallis, OR, 97331 USA e-mail: deleenhp@math.oregonstate.edu.

T. The boundary condition corresponds to a lethal or absorbing boundary, where fish cannot survive. Other boundary conditions (e.g. no-flux or mixed-type) can be handled with a similar approach. The fish diffuse with diffusion constant $D > 0$, are recruited at rate $R > 0$, die at per capita rate $\mu > 0$ and are harvested at per capita rate $H(X)$ which depends on the location X. We note that this model does not include any density-dependent features, since recruitment occurs at a constant rate R in space and time, and is independent of the current fish density U. This scenario is motivated by reef fish whose habitats are restricted to specific reef patches. The boundary of such a patch is lethal, possibly due to the presence of a predator patrolling the patch boundary. Recruitment happens after larvae have settled in the patch. The assumption of a uniform recruitment rate corresponds to a case where adult fish abundantly generate larvae over many reef patches, which in turn are dispersed over these patches by diffusion and/or advection due to ocean currents. Since the fishermen's fleet is limited, we assume that H takes values in the interval $[0, \bar{H}]$, where $\bar{H} > 0$ denotes the maximal harvesting rate. In what follows $< F >:= 1/L \int_{-L/2}^{L/2} F(X)dX$ denotes the average of a function $F(X)$, defined on the interval $[-L/2, L/2]$. The problem addressed here is to find the function $H(X)$ which maximizes the steady state functional:

$$
J(H(X)) = < H(X)U(X) > + Q < U(X) >,
\tag{2}
$$

where $Q \geq 0$ is a fixed weight parameter, and $U(X)$ is a steady state of (1) using $H(X)$. This functional reflects the tradeoff between $< HU >$, the average harvest and the average fish density, weighted by a parameter Q which is small in regions whenever there is little pressure by conservationists to limit fishing, and large otherwise. Several model parameters can be scaled out, yielding the scaled model:

$$
\begin{aligned}
u_t &= u_{xx} - (1 + h(x))\, u + 1, \quad -l/2 < x < l/2, \\
u(-l/2, t) &= u(l/2, t) = 0, \quad \text{for all } t \geq 0
\end{aligned}
\tag{3}
$$

with scaled functional:

$$
j(h(x)) = < h(x)u(x) > + q < u(x) >,
\tag{4}
$$

which needs to be maximized over functions $h(x)$ taking values in $[0, \bar{h}]$, and where $u(x)$ is the steady state of (3) corresponding to $h(x)$. Averages appearing in the scaled functional, are averages over the scaled interval $[-l/2, l/2]$. The scaled problem contains only 3 parameters: the weight parameter $q \geq 0$, the coastal length $l > 0$, and the maximum

harvesting rate $\bar{h} > 0$. The main results will be phrased in terms of these parameters, but they are easily translated in terms of the parameters of the unscaled problem. By letting $v = u'$ where $'$ denotes d/dx, we recast the steady state problem associated to (3) as:

$$u' = v \tag{5}$$
$$v' = (1 + h(x))\, u - 1 \tag{6}$$
$$u(-l/2) = u(l/2) = 0 \tag{7}$$

The problem is to find a measurable function $h(x)$, taking values in the interval $[0, \bar{h}]$ for x in $[-l/2, l/2]$ a.e., such that for this particular choice of $h(x)$, a solution $(u(x), v(x))$ to $(5) - (6)$ exists that satisfies the boundary condition (7), and the constraint that $u(x) \geq 0$ for all $x \in [-l/2, l/2]$ (fish densities are non-negative). Standard existence results [4], [1] yield:

Theorem 1. *There exists an admissible control $h^*(x)$ defined for $x \in [-l/2, l/2]$, which maximizes the scaled functional (4).*

The Hamiltonian associated to $(5) - (6)$ and functional (4) is:

$$H(u, v, \lambda_1, \lambda_2, h) = \frac{1}{l}(h + q)u + \lambda_1 v + \lambda_2 \left((1 + h)\, u - 1\right), \tag{8}$$

where (λ_1, λ_2) are the adjoint variables. By Pontryagin's maximum principle [4], any maximum for the functional attained at some $(u^*(x), v^*(x), \lambda_1^*(x), \lambda_2^*(x), h^*(x))$ must maximize the Hamiltonian with respect to $h(x)$:

$$H(u^*(x), v^*(x), \lambda_1^*(x), \lambda_2^*(x), h(x)) \leq$$
$$H(u^*(x), v^*(x), \lambda_1^*(x), \lambda_2^*(x), h^*(x)), \tag{9}$$

for all x in $[-l/2, l/2]$ and $h(x)$ in $[0, \bar{h}]$, and must solve the Hamiltonian system

$$u' = \partial H/\partial \lambda_1 = v \tag{10}$$
$$v' = \partial H/\partial \lambda_2 = (1 + h)u - 1 \tag{11}$$
$$\lambda_1' = -\partial H/\partial u = -(h + 1)\lambda_2 - \frac{h + q}{l} \tag{12}$$
$$\lambda_2' = -\partial H/\partial v = -\lambda_1 \tag{13}$$

with boundary and transversality conditions:

$$u(-l/2) = u(l/2) = 0 \tag{14}$$
$$\lambda_2(-l/2) = \lambda_2(l/2) = 0 \tag{15}$$

Since H is linear in the control variable h, it follows from (9) that

$$h^*(x) = \begin{cases} 0, & \text{if } u^*(x)(1/l + \lambda_2^*(x)) < 0 \\ \bar{h}, & \text{if } u^*(x)(1/l + \lambda_2^*(x)) > 0 \end{cases} \tag{16}$$

The set of points $\{(u, v, \lambda_1, \lambda_2) | u = 0 \text{ or } \lambda_2 = -1/l\}$ is called the *switching surface* of the Hamiltonian system. It can be shown that an optimal solution $(u^*(x), v^*(x), \lambda_1^*(x), \lambda_2^*(x))$ cannot belong to the part of the switching surface where $u = 0$, other than at the initial and final locations $x = \pm l/2$. Notice that this fact, combined with (16) and the transversality condition (15), also shows that

$h^*(x) = \bar{h}$ for all x near $x = -l/2$ and $x = l/2$. In addition, since $u^* > 0$ in $(-l/2, l/2)$, the optimal control $h^*(x)$ takes the form:

$$h^*(x) = \begin{cases} 0, & \text{if } \lambda_2^*(x) < -1/l \\ \bar{h}, & \text{if } \lambda_2^*(x) > -1/l \end{cases} \tag{17}$$

The question is whether the state $(u^*(x), v^*(x), \lambda_1^*(x), \lambda_2^*(x))$ of the Hamiltonian system ever crosses, or remains on the (smaller) switching surface

$$\mathcal{S} = \{(u, v, \lambda_1, \lambda_2) | \lambda_2 = -1/l\} \tag{18}$$

If the state remains on $\mathcal{S}$ for x in some subinterval of $[-l/2, l/2]$, then the value of $h^*(x)$ is not determined by (16). The control is then said to be *singular* and a more detailed analysis would be required to determine $h^*(x)$. However, it can be shown that optimal controls cannot be singular. If an optimal control requires a switch from $\bar{h}$ to 0, and hence a crossing of $\mathcal{S}$, then it must necessarily switch back to $\bar{h}$ at least once later during in the interval $[-l/2, l/2]$, since it must equal $\bar{h}$ for all x near $-l/2$. We will calculate a bound for q, such that below this bound, there is no switch. If, on the other hand, q exceeds this bound, no switches occur if l falls below some threshold, and exactly two switches occur when l is above it.

If $h(x) = \bar{h}$ or $h = 0$ for all x, then the adjoint system is a linear time-invariant system of the form

$$\lambda_1' = -a\lambda_2 - \frac{b}{l} \tag{19}$$
$$\lambda_2' = -\lambda_1 \tag{20}$$

for suitable $a > 0$ and $b > 0$.

Lemma 1. *System $(19) - (20)$ has a unique equilibrium point $E = (0, -b/(al))$ which is a saddle. The stable and unstable manifold have slope $1/\sqrt{a}$ and $-1/\sqrt{a}$ respectively.*

A. The case $0 < q \leq 1$.

It is first shown that switches in the value of $h(x)$ are not possible in this case. Consider Figures 1 and 2, which depict some orbits of the adjoint system $(12) - (13)$ when $h(x) = \bar{h}$ and 0 respectively. Notice that the steady state has coordinates $(0, -(h + q)/(h + 1)l)$, and thus it does not lie below the switching line $\{(\lambda_1, \lambda_2) | \lambda_2 = -1/l\}$. The problem is to determine whether or not there are solutions starting on the λ_1-axis at $x = -l/2$ which reach the horizontal switching line $\{(\lambda_1, \lambda_2) | \lambda_2 = -1/l\}$ at some $x_s < l/2$. As it turns out, there are no such solutions, and a proof is briefly sketched next: (i) If $\lambda_1(-l/2) \leq 0$, this is impossible, as the solution will remain in the second quadrant because it is forward invariant. Notice also that the transversality (15) at $x = l/2$ cannot hold for such solution. (ii) If $0 < \lambda_1(-l/2) \leq \lambda_s$, where $\lambda_s := (\bar{h} + q)/(\sqrt{\bar{h} + 1}\, l)$ is the intercept of the stable manifold (the straight line with positive slope -in red- in Figure 1), this is also impossible. Indeed, this follows because the region that lies above the stable and unstable manifold (the straight line with negative slope -in red- in Figure 1) is forward invariant, and because the lowest point of this region -the steady state- does not lie below the switching line. (iii) If

$\lambda_1(-l/2) > \lambda_s$, the solution may reach the switching line at some $x_s < l/2$. Assume it happens and denote the state of the adjoint system at $x = x_s$ by $(\lambda_1(x_s), -1/l)$. Note that necessarily $\lambda_1(x_s) \geq 0$ because the region which is part of the fourth quadrant which lies below the stable manifold and above the unstable manifold, is forward invariant. When the solution reaches the switching line, it will cross it, and thus the control variable h now switches from $\bar{h}$ to 0. Thus the adjoint system becomes $(12) - (13)$ but now with $h(x) = 0$, whose orbits are depicted in Figure 2. From the orbits it is clear that for all $x > x_s$, the solution will remain below the switching line $\lambda_2 = -1/l$. This follows from the fact that the region $\{(\lambda_1, \lambda_2) | \lambda_1 \geq 0, \lambda_2 \leq -1/l\}$ is forward invariant. Thus, the possibility that an optimal control exhibits a switch, has been ruled out.
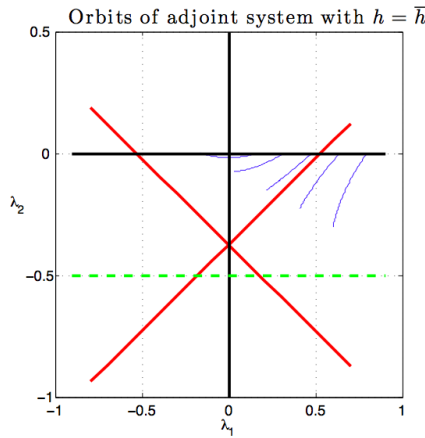

Fig. 1.   Phase portrait of system $(12) - (13)$, with $l = 2$, $q = 0.5$ and $h = \bar{h} = 1$. Stable manifolds are the straight lines with positive slope (in red), whereas unstable manifolds are straight lines with negative slope (also in red). The dashed curve (in green) is the switching line where $\lambda_2 = -1/l$.
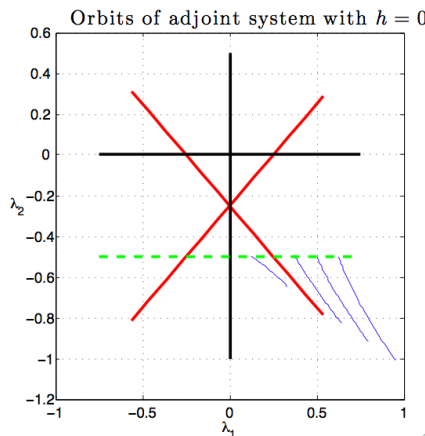

Fig. 2. Phase portrait of $(12) - (13)$, $l = 2$, $q = 0.5$ and $h = \bar{h} = 0$.

The foregoing discussion shows that every solution of the adjoint system $(12) - (13)$ with $h(x) = \bar{h}$ which satisfies the transversality conditions (15), must be such that $0 < \lambda_1(-l/2) < \lambda_s$. Such a solution always exists, and is unique. Indeed, solving the adjoint system with $h(x) = \bar{h}$ and initial condition $(\lambda_0, 0)$ at $x = -l/2$, where the parameter λ_0 takes

values in the interval $(0, \lambda_s)$:

$$\lambda_1(x) = -\lambda_s \frac{\sinh\left(\sqrt{\bar{h}+1}(x + l/2 - \beta)\right)}{\cosh\left(\sqrt{\bar{h}+1}\beta\right)}$$

$$\lambda_2(x) = \frac{\lambda_s}{\sqrt{\bar{h}+1}}\left[\frac{\cosh\left(\sqrt{\bar{h}+1}(x + l/2 - \beta)\right)}{\cosh\left(\sqrt{\bar{h}+1}\beta\right)} - 1\right], \quad (21)$$

where β is uniquely defined by $\tanh\left(\sqrt{\bar{h}+1}\beta\right) = \frac{\lambda_0}{\lambda_s}$. Let $T > -l/2$ denote the location where the solution reaches the λ_1-axis again. Then $\lambda_2(T) = 0$, and thus

$$\cosh\left(\sqrt{\bar{h}+1}(T + l/2 - \beta)\right) = \cosh\left(\sqrt{\bar{h}+1}\beta\right)$$

or writing T explicitly as a function of λ_0 using β:

$$T(\lambda_0) = \frac{2}{\sqrt{\bar{h}+1}}\text{arctanh}\left(\frac{\lambda_0}{\lambda_s}\right) - \frac{l}{2}$$

Notice that $\lim_{\lambda_0 \to 0} T = -l/2$, $\lim_{\lambda_0 \to \lambda_s} T = +\infty$ and T is increasing. Hence, there is a unique λ_0^* such that

$$T(\lambda_0^*) = l/2. \quad (22)$$

Plugging $\lambda_0 = \lambda_0^*$ in (21) and β yields the unique corresponding $(\lambda_1^*(x), \lambda_2^*(x))$ components of the solution of the Hamiltonian system $(10) - (13)$ that satisfy the boundary conditions (15).

The (u, v) components corresponding to an optimal solution of the Hamiltonian system $(10) - (13)$ when $h(x) = \bar{h}$ for all x in $[-l/2, l/2]$ can now be determined as well. Some orbits of $(10) - (11)$ are depicted in Figure 3. Arguing as was done for the adjoint system, it is not hard to show that the only possible solutions of $(10) - (11)$ with $h(x) = \bar{h}$ satisfying (14) must be such that the initial condition $(0, v_0)$ at $x = -l/2$ is such that $0 < v_0 < 1/\sqrt{\bar{h}+1}$. This is because if $v_0 \leq 0$ or if $v_0 \geq 1/\sqrt{\bar{h}+1}$, then the boundary condition (14) at $x = l/2$ cannot be satisfied. If $0 < v_0 < 1/\sqrt{\bar{h}+1}$, the solution is given by:

$$u(x) = -\frac{v_0 \cosh\left[\sqrt{\bar{h}+1}(x + l/2 - \alpha)\right]}{\sqrt{\bar{h}+1}\sinh\left(\sqrt{\bar{h}+1}\alpha\right)} + \frac{1}{\bar{h}+1}$$

$$v(x) = -v_0 \frac{\sinh\left[\sqrt{\bar{h}+1}(x + l/2 - \alpha)\right]}{\sinh\left(\sqrt{\bar{h}+1}\alpha\right)}, \quad (23)$$

where α is uniquely defined by

$$\text{cotanh}\left(\sqrt{\bar{h}+1}\alpha\right) = \frac{1}{v_0\sqrt{\bar{h}+1}} \quad (24)$$

Let $T_0 > -l/2$ be such that $u(T_0) = 0$, then by (23) and (24)

$$\cosh\left[\sqrt{\bar{h}+1}(T_0 + l/2 - \alpha)\right] = \cosh\left[\sqrt{\bar{h}+1}\alpha\right],$$

or, since $T_0 > -l/2$, that $T_0 = 2\alpha - l/2$. Using (24) once more, T_0 can be written explicitly as a function of v_0:

$$T_0(v_0) = \frac{2}{\sqrt{\bar{h}+1}}\text{arccoth}\left(\frac{1}{v_0\sqrt{\bar{h}+1}}\right) - \frac{l}{2} \quad (25)$$

Notice that $\lim_{v_0 \to 0} T_0 = -l/2$, $\lim_{v_0 \to 1/\sqrt{\bar{h}+1}} T_0 = +\infty$ and T_0 is increasing. Hence, there is a unique v_0^* such that $T_0(v_0^*) = l/2$, namely

$$v_0^* = \frac{1}{\sqrt{\bar{h}+1} \coth\left(\sqrt{\bar{h}+1}\, l/2\right)}. \qquad (26)$$

Plugging $v_0 = v_0^*$ in (23) and (24) yields the unique corresponding $(u^*(x), v^*(x))$ components of the solution of the Hamiltonian system $(10) - (13)$ that satisfy the boundary conditions (14). In summary,
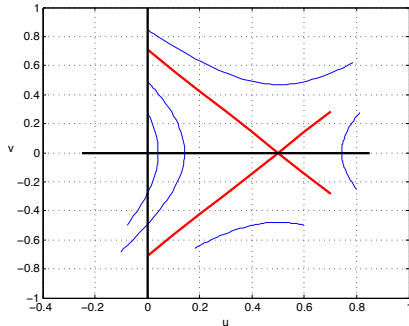

Fig. 3. Phase portrait of $(10) - (11)$, $l = 2$, $q = 0.5$ and $h = \bar{h} = 1$.

Theorem 2. *If $0 < q \le 1$, then there is a unique optimal control $h^*(x) = \bar{h}$ for all x in $[-l/2, l/2]$ which maximizes the scaled functional (4) for the steady state problem $(5)-(7)$. The corresponding optimal fish density $u(x) = u^*(x)$ is given by $(23) - (24)$ with $v_0 = v_0^*$, where v_0^* is defined in (26).*

B. The case $q > 1$.

Some orbits of the adjoint system $(12)-(13)$ are depicted in Figures 4 and 5, using $h(x) = \bar{h}$ and 0 respectively. In view of (17), solutions of the adjoint system follow orbits of Figure 4 as long as $\lambda_2 > -1/l$, and those of Figure 5 whenever $\lambda_2 < -1/l$.
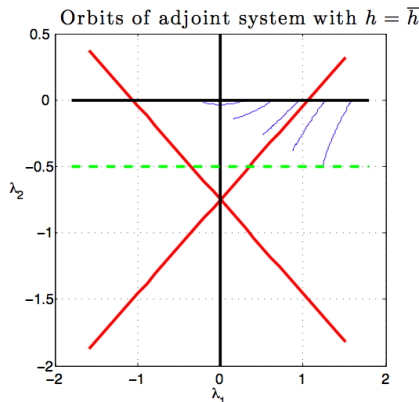

Fig. 4. Phase portrait of $(12) - (13)$, $l = 2$, $q = 2$ and $h = \bar{h} = 1$.

Defining F_1 and F_2 as the (autonomous) vector field of the adjoint system $(12) - (13)$ when $h(x) = \bar{h}$ and $h(x) = 0$
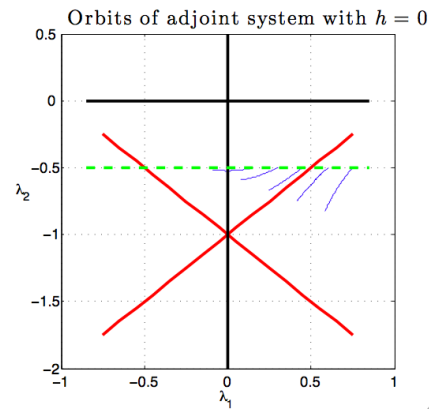

Fig. 5. Phase portrait of $(12) - (13)$, $l = 2$, $q = 2$ and $h = 0$.

respectively, the adjoint system can be rewritten as an autonomous system:

$$\dot{\lambda} = \begin{cases} F_1(\lambda), & \text{if } \lambda \in \mathcal{S}_a \\ F_2(\lambda), & \text{if } \lambda \in \mathcal{S}_b \end{cases} \qquad (27)$$

where $\mathcal{S}_a := \{(\lambda_1, \lambda_2) | \lambda_2 > -1/l\}$ and $\mathcal{S}_b := \{(\lambda_1, \lambda_2) | \lambda_2 < -1/l\}$ are the regions above and below the switching line respectively. The objective is to find solutions of (27) that satisfy (15). Actually, it can be shown that thanks to a symmetry property of the adjoint system, it suffices to consider solutions defined on just $[-l/2, 0]$ (half of the control horizon $[-l/2, l/2]$) that start on the λ_1-axis at $x = -l/2$ and end on the λ_2-axis at $x = 0$. Consequently, solutions of (27) satisfying

$$\lambda_2(-l/2) = \lambda_1(0) = 0 \qquad (28)$$

should be determined. A brief sketch to approach this problem is given next. Consider all solutions of (27) starting on the positive λ_1-axis at $x = 0$ (rather than at $x = -l/2$; this is easily achieved by a shift in x), by parametrizing them by the λ_1-coordinate of their initial condition. Concerning these solutions, focus on the following two questions: 1. Which solutions reach the λ_2-axis? 2. For those solutions reaching the λ_2-axis, what is the first x-value larger than 0 for which this happens? With respect to the first question, it will be seen that only some solutions, namely those with corresponding parameter values that are not too high, reach the λ_2-axis. Moreover, some -but not all- of these solutions cross the switching line. With respect to the second question, it will be seen that the first value of x for which solutions reach the λ_2-axis, is an increasing function of the parameter. It increases from zero to infinity, and hence there will be a unique solution for which it equals $l/2$. The solution corresponding to this x-value, is the sought-after solution to problem (27) with (28). Formally, we consider

$$\dot{\lambda} = \begin{cases} F_1(\lambda), & \text{if } \lambda \in \mathcal{S}_a \\ F_2(\lambda), & \text{if } \lambda \in \mathcal{S}_b \end{cases}, \begin{pmatrix} \lambda_1(0) \\ \lambda_2(0) \end{pmatrix} = \begin{pmatrix} \lambda_0 \\ 0 \end{pmatrix} \qquad (29)$$

where $\lambda_0 > 0$ is a parameter. Define the positive constants $a_1 = \bar{h}+1$, $b_1 = \bar{h}+q$ and $a_2 = 1$, $b_2 = q$, so that the vector fields F_i, $i = 1, 2$, can be rewritten as $F_i(\lambda_1, \lambda_2) = (-a_i\lambda_2 -$

$\frac{b_i}{l}, -\lambda_1)^T$. Also define λ_1-coordinates of the intercepts of the stable manifolds of the adjoint system $(12)-(13)$ with $h(x) = \bar{h}$, and with $h(x) = 0$ and the λ_1-axis as

$$i_1 := \frac{b_1}{\sqrt{a_1 l}} \text{ and } i_2 := \frac{b_2}{\sqrt{a_2 l}} \tag{30}$$

respectively. Since $q > 1$, it can be verified by simple calculations that

$$e_1 := -\frac{b_1}{a_1 l} > -\frac{b_2}{a_2 l} =: e_2, \text{ and}$$

$$i_{s_1} := \frac{\sqrt{a_1}}{l}\left(\frac{b_1}{a_1} - 1\right) < \frac{\sqrt{a_2}}{l}\left(\frac{b_2}{a_2} - 1\right) =: i_{s_2} \tag{31}$$

The first inequality in (31) expresses that the λ_2-coordinate of the equilibrium point of the adjoint system $(12) - (13)$ with $h(x) = \bar{h}$ is larger than the λ_2-coordinate of the equilibrium point of the adjoint system $(12) - (13)$ with $h(x) = 0$. The second inequality expresses that the λ_1-coordinate of the intersection of the switching line where $\lambda_2 = -1/l$ and the stable manifold of the equilibrium point of the adjoint system $(12) - (13)$ with $h(x) = \bar{h}$, is smaller than the λ_1-coordinate of the stable manifold of the equilibrium point of the adjoint system $(12) - (13)$ with $h(x) = 0$. These geometrical observations are illustrated in Figures 4 and 5. We define two important values for the parameter λ_0:

$$\lambda_0^* = \frac{\sqrt{2b_1 - a_1}}{l} \text{ and } \lambda_0^{**} = \left((\lambda_0^*)^2 + i_{s,2}^2\right)^{1/2} \tag{32}$$

Let $\lambda(x)$, $x > 0$, be the (forward) solution of (29), and define

$$T_0(\lambda_0) = \inf\{x > 0 | \lambda_1(x) = 0\}, \tag{33}$$

the first instance where $\lambda(x)$ hits the λ_2-axis. If $\lambda(x)$ never hits the λ_2-axis, then set $T(\lambda_0) = +\infty$. Since the system is piecewise linear, it can be solved analytically, and thus T_0 can be calculated explicitly.

Theorem 3. $T_0 : (0, +\infty) \to (0, +\infty]$ *is continuous and increasing, and* $\lim_{\lambda_0 \to 0} T_0(\lambda_0) = 0$ *and* $\lim_{\lambda_0 \to \lambda_0^{**}} T_0(\lambda_0) = +\infty$. *There exists a unique* $\bar{\lambda}_0 \in (0, \lambda_0^{**})$ *such that* $T_0(\bar{\lambda}_0) = l/2$ *with corresponding solution* $\lambda(x)$ *of (29), satisfying* $\lambda_2(0) = \lambda_1(l/2) = 0$ *This solution hits the switching line where* $\lambda_2 = -1/l$ *for some* $x \in (0, l/2)$ *if and only if* $\lambda_0^* < \bar{\lambda}_0$, *or equivalently, by applying the increasing function* T_0, *if and only if*

$$\frac{1}{\sqrt{a_1}} \text{arctanh}\left(\frac{\lambda_0^*}{i_1}\right) < \frac{l}{2} \tag{34}$$

Inequality (34) is of great practical relevance. It determines the minimal coastal length

$$l_{\min} := \frac{2}{\sqrt{a_1}} \text{arctanh}\left(\frac{\lambda_0^*}{i_1}\right) =$$

$$\frac{2}{\sqrt{\bar{h} + 1}} \text{arctanh}\left(\frac{\sqrt{(\bar{h} + 1)(\bar{h} + 2q - 1)}}{\bar{h} + q}\right)$$

that is required in order for the implementation of an MPA to be optimal. For coastlines with a length below $l_{\min}$, MPAs should not be installed, and fishing should take place with maximal harvesting rate $\bar{h}$ everywhere along the coast. For coastlines longer than $l_{\min}$, the optimal solution requires the

placement of an MPA. Where this should occur is addressed in the next result

Theorem 4. *Assume that* $l > l_{\min}$, *and consider the unique value* $\bar{\lambda}_0 \in (\lambda_0^*, \lambda_0^{**})$ *defined in Theorem 3. Denote the corresponding solution of (29) by* $\lambda(x)$, *and let* $T_s(\bar{\lambda}_0)$ *be the x-value at which the solution* $\lambda(x)$ *hits the switching line where* $\lambda_2 = -1/l$. *Then* $\sqrt{a_1} T_s(\bar{\lambda}_0)$ *equals*

$$\text{arctanh}\left(\frac{\bar{\lambda}_0}{i_1}\right) - \text{arccosh}\left(\frac{b_1/a_1 - 1}{b_1/a_1}\frac{1}{\sqrt{1 - \left(\frac{\bar{\lambda}_0}{i_1}\right)^2}}\right) \tag{35}$$

if $\lambda_0^* < \bar{\lambda}_0 < i_1$, *and*

$$\text{arccoth}\left(\frac{\bar{\lambda}_0}{i_1}\right) - \text{arcsinh}\left(\frac{b_1/a_1 - 1}{b_1/a_1}\frac{1}{\sqrt{\left(\frac{\bar{\lambda}_0}{i_1}\right)^2 - 1}}\right) \tag{36}$$

if $i_1 \leq \lambda_0^* < \lambda_0^{**}$.

The main result combines Theorems 3 and 4:

Theorem 5. *Assume that* $q > 1$. *If* $l \leq l_{\min}$, *then there is a unique optimal control* $h^*(x) = \bar{h}$ *for all* x *in* $[-l/2, l/2]$ *which maximizes the scaled functional (4) for the steady state problem* $(5) - (7)$. *The corresponding optimal fish density* $u(x) = u^*(x)$ *is given by* $(23) - (24)$ *with* $v_0 = v_0^*$, *where* v_0^* *is defined in (26). If* $l > l_{\min}$, *then there is a unique optimal control*

$$h^*(x) = \begin{cases} 0, & \text{if } x \in [-l/2 + T_s(\bar{\lambda}_0), l/2 - T_s(\bar{\lambda}_0)] \\ \bar{h}, & \text{otherwise} \end{cases}$$

where $T_s(\bar{\lambda}_0)$ *is defined in Theorem 4. This optimal control maximizes the scaled functional (4) for the steady state problem* $(5) - (7)$. *There is a corresponding optimal fish density* $u^*(x)$, *defined as the u-component of the unique solution to* $(5) - (7)$, *with* $h(x) = h^*(x)$.

Stability of the optimal steady state A natural question is whether the steady state corresponding to an optimal control $h^*(x)$ is asymptotically stable for (3). Linearization yields an eigenvalue problem:

$$\begin{aligned} \lambda w &= w_{xx} - (1 + h^*(x))w \\ w(-l/2) &= w(l/2) = 0 \end{aligned} \tag{37}$$

The operator $L[w] := w_{xx} - (1 + h^*(x))w$ is self-adjoint with respect to the inner product $(w_1, w_2) := \int_{-l/2}^{l/2} w_1 w_2 dx$, hence all eigenvalues λ are real. For any eigenvalue-eigenfunction pair $(\lambda, w(x))$, an integration by parts yields:

$$\lambda \int_{-l/2}^{l/2} w^2 dx = -\int_{-l/2}^{l/2} w_x^2 + (1 + h^*(x))w^2 dx,$$

from which $\lambda < 0$, providing evidence for local stability of the optimal steady state.

Limited circulation. For review only

REFERENCES

[1] L. D. Berkovitz, Existence Theory for Optimal Control Problems. In Optimal Control and Differential Equations. Eds: A.B. Schwarzkopf, W.G. Kelley, and S.B. Eliason. Academic Press 1978.

[2] P. De Leenheer, Optimal placement of Marine Protected Areas, http://arxiv.org/submit/753214.

[3] L.C. Evans. An Introduction to Mathematical Optimal Control Theory. Version 0.2. Download from http://math.berkeley.edu/ēvans/control.course.pdf.

[4] E.B. Lee, and L. Markus, Foundations of optimal control theory. SIAM, New York, 1967.

[5] D. Liberzon. Calculus of variations and optimal control theory, Princeton University Press, 2011.

[6] M.G. Neubert, Marine reserves and optimal harvesting, Ecology Letters 6, 843-849, 2003.

[7] J.W. White et al, Linking models with monitoring data for assessing performance of no-take marine reserves, Frontiers in Ecology and the Environment 9(7), 390-399, 2011.