Homework assignment 1*

Due date: Wednesday September 12, 2007

1. Consider the population model again. Fix $f = 1$ and let $m \geq 0$ be arbitrary.

   (a) Show that the system is unstable and thus almost all solutions grow unbounded.

   (b) Let's modify the model to reflect death of adults. Assume that all juveniles mature between two consecutive censuses (so $f = 1$ in terms of our original model). Reproduction, and right after that death, take place at the end of one cycle, right before the census is taken. Only matures that were mature at the beginning of the cycle are capable of reproduction. Thus, the juveniles that matured in this cycle don't reproduce yet. Right after reproduction some of the adults die and a fraction s survives to the census. This census is the starting point of the next cycle. How does the original model change? (Still assume that $m \geq 0$ is arbitrary.) Give conditions for the parameters such that the system is stable, asymptotically stable and unstable respectively. Show that the condition for asymptotic stability implies that $m < 1$. Interpret this inequality.

2. Suppose that at time t you possess a certain capital $C(t)$ and some investment $I(t)$. At the start of a new investment cycle, only capital can be invested (there is no re-investment). You can invest a fraction $f \in [0, 1]$ of your capital, and thus $(1 - f)C(t)$ is the capital you keep. By the end of the financial cycle the invested fraction of the capital yields a guaranteed interest of $r\%$. However, some of the returns are delinquent, and only a fraction d of the value of the investment returns as capital to you.

   (a) Construct a model whose state consists of $C(t)$ and $I(t)$.

   (b) Let $f = 1$. Show that you will go broke if $(1 + r)d < 1$, and that you will make money if the inequality is reversed. Interpret these inequalities.

   (c) Let $f = 0.5$. How do the above inequalities change? Interpret your result.

   (d) Generalize this model by diversifying your investment. That is, assume there are n types of investments $I_1, \ldots, I_n$ each having an interest of r_i and a delinquency factor of d_i.

3. (a) Calculate the Page Rank assuming that the World Wide Web consists of 3 websites. Suppose that the websites have no self-links, and that the WWW is strongly connected (that is, given any of the websites, it is possible to reach any of the other websites following one or more links). What configuration yields the highest page rank for page 1?

   (b) Check your calculations with MATLAB, see remark below. Print a copy of you MATLAB worksheet and turn it in together with your homework.

4. Consider the continuous time linear system

$$\dot{x} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} x.$$

   (a) Determine a fundamental matrix solution for this sytem.

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(b) Determine the solution $x(t)$ of this system when the initial condition at $t = 0$ is

$$x(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

5. **Optional problem; requires some background in linear algebra**.

Consider the continuous time linear system

$$\dot{x} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix} x,$$

where $x \in \mathbb{R}^n$. [Notice that you can write the system matrix as a product of a column vector and a row vector as follows:

$$\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & \dots & 1 \end{pmatrix}]$$

Determine a fundamental matrix solution for this system. (**Hint: Think geometrically to find eigenvalue-eigenvector pairs**)

Remark 1. MATLAB is available on all campus computers. It is software that is very friendly to matrices. When you start it up, you will see a command line >> appear. To define a matrix

$$\begin{pmatrix} 1 & 1.2 \\ 3 & 4 \end{pmatrix},$$

type

$$A = [1 \;\; 1.2 \; ; \; 3 \;\; 4]$$

after the command line and hit return. MATLAB will produce an output in the form of the desired matrix. Now that this matrix has been defined, you can calculate its eigenvector-eigenvalues pairs by typing the following:

[T,D]=eig(A)

MATLAB will return two matrices T and D, where T contains the eigenvectors and D is a diagonal matrix containing the eigenvalues on the diagonal.