Homework assignment 2*

Due date: Wednesday October 10, 2007

1. Let v be a nonzero vector in $\mathbb{R}^3$ and consider the linear system

$$\dot{x} = v \times x,$$

where $\times$ stand for cross product (see your old calc book in case you forgot). This equation describes the motion of a rigid body rotating around an axis with direction given by the vector v.

   (a) Write the system in the usual form $\dot{x} = Ax$, and find the general solution.

   (b) Is the system stable, asymptotically stable or unstable?

2. In class we discussed a bioreactor model with constant input nutrient concentration S^0 (we actually set $S^0 = 1$) and variable dilution rate $D > 0$. In this problem, we do things the other way round: We will set $D = 1$ and let $S^0 > 0$ be variable. The resulting equations are:

$$\begin{aligned} \dot{S} &= S^0 - S - Sx \\ \dot{x} &= x(S-1) \end{aligned}$$

   (a) Perform phase plane analysis like we did in class, by determining the nullclines, all equilibria and the direction of the vector field in the various regions of the non-negative quadrant of $\mathbb{R}^2$ (use NW,SW etc arrows).

   Based on the resulting sketch, what do you think happens to solutions when $t \to \infty$?

   (b) Linearize the system at the equilibria and discuss the nature of the linearization (stable node, center etc).

3. Consider

$$\begin{aligned} \dot{x} &= -y + dx(x^2 + y^2) \\ \dot{y} &= x + dy(x^2 + y^2), \end{aligned}$$

where d is a real parameter. Determine stability of the equilibrium at $(0, 0)$ in terms of d. (**Hint: Use polar coordinates.**)

4. Determine stability of the zero solution of the following equations:

   - $\ddot{x} + \dot{x} + \sin x = 0$.
   - $\ddot{x} + \dot{x}\cos x + \sin x = 0$.

Note that these are problems 12.5 # 13 and # 14 from our text. Ignore the hint given there, and instead try the following function:

$$V(x, y) = 2\sin^2\left(\frac{x}{2}\right) + \frac{y^2}{2}.$$

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