Homework assignment 4*

Due date: Monday December 3, 2007

1. **Buckling of a tower** See figure 8.16 on p. 501 in the text.

   Consider the following Cauchy-Euler equation[1] which describes the deflection $y(x)$ from the vertical in terms of the height x measured from the top of the untruncated tower:

$$\begin{aligned} x^2y'' + \frac{Pa^2}{EI}y &= 0, \quad x \in (a, a+L) \\ y(a) &= 0 \\ y'(a+L) &= 0 \end{aligned}$$

   The top of the tower corresponds to $x = a$ (a is the part of the tower that is truncated) and the ground corresponds to $x = a+L$, so the height of the tower is L. The positive parameters are as follows: P for the vertical load, I for the moment of inertia, E for the modulus of elasticity.

   Under certain conditions -in particular when the load is high enough- the tower may *buckle* and the purpose of this problem is to determine when this will happen.

   By definition, buckling occurs if the above boundary value problem has a nontrivial solution ($y(x) \neq 0$).

   (a) Show that if $P \leq EI/(4a^2)$, then there is no nontrivial solution which implies that no buckling occurs. Thus, for small enough loads there is no buckling.

   (b) Assume henceforth that $P > EI/(4a^2)$. In that case there *may be* a nontrivial periodic solution as we will discover soon. First show that the first boundary condition implies that a nontrivial solution must be of the form:

$$y(x) = c\sqrt{x}\sin\left(\beta \ln(x/a)\right),$$

   where c is a constant to be determined later and

$$\beta = \sqrt{\frac{Pa^2}{EI} - \frac{1}{4}}$$

   is a positive constant depending on the load and various parameters. The second boundary condition implies that:

$$c\left(\tan\left(\alpha\beta\right) + 2\beta\right) = 0,$$

   where

$$\alpha = \ln\left(\frac{a+L}{a}\right).$$

   This implies that either $c = 0$ (in which case no buckling occurs), or that

$$\tan\left(\alpha\beta\right) = -2\beta.$$

   In terms of β, graph the above tangent function and linear function, and show that they intersect infinitely many times for $\beta > 0$. For each β at which these graphs intersect (don't intersect), buckling occurs (does not occur)

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[1] Here, it is more convenient to assume that solutions have the form $y(x) = \left(\frac{x}{a}\right)^r$ for values r satisfying the indicial equation, instead of $y(x) = x^r$.

(c) Show that the first time the above graphs intersect happens at some critical value of β which we denote as $\beta_c > 0$, and show that β_c belongs to the interval (β_0, β_1), where

$$\beta_0 = \frac{\pi}{2\alpha}, \quad \beta_1 = \frac{\pi}{\alpha}.$$

Express the *critical load* P_c, which is the minimal load for which buckling occurs, in terms of β_c.

2. Determine all real eigenvalues and eigenfunctions of

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(\pi) - ay'(\pi) = 0,$$

for all possible values of the positive parameter $a > 0$.

3. Consider the linear operator

$$L[y] = y^{(4)},$$

defined on the set of functions having continuous derivatives up to order 4, that satisfy the following boundary conditions:

$$y(a) = y'(a) = y(b) = y'(b) = 0.$$

Show that L is self-adjoint, ie that for all y_1 and y_2 in the domain of L, we have that:

$$(y_1, L[y_2]) = (L[y_1], y_2).$$

Use this to prove that eigenfunctions of the problem:

$$L[y] + \lambda y = 0, \quad y(a) = y'(a) = y(b) = y'(b) = 0$$

corresponding to distinct eigenvalues are orthogonal, ie that

$$(y_1, y_2) = 0,$$

whenever y_1 and y_2 are eigenfunctions corresponding to distinct eigenvalues.

4. The following equation describes the displacement of an elastic beam which is clamped on both ends $x = 0$ and $x = 1$:

$$y^{(4)} + \lambda y = 0, \quad x \in (0, 1), \quad y(0) = y'(0) = y(1) = y'(1) = 0.$$

It can be shown (no need to do this) that the eigenvalues λ are non-positive. Determine all eigenvalues and eigenfunctions.