HOMEWORK SOLUTIONS MTH 228

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1. Homework 1, due 09/29

Do the following exercises from [1] 19.1g and m, 19.5e, 19.11, 19.15, 20.4, 20.6.

1.1. **19.1 (g).** Find the derivative of $f(x) = \sin(5x)$.
Solution: In this exercise if we let $h(x) = 5x$ and $g(x) = \sin(x)$, notice that $f(x) = (g \circ h)(x)$. Using the chain rule we know that $f' = g'(h)\, h'$, in other words

$$\frac{d}{dx}(f(x)) = \frac{d}{dx}(\sin(5x)) = \cos(5x)\frac{d}{dx}(5x) = 5\cos(5x)\,.$$

1.2. **19.1 (m).** Find the derivative of $f(x) = \ln(\sin^2(x))$.
Solution: In this exercise if we let $j(x) = \sin(x)$, $h(x) = x^2$ and $g(x) = \ln(x)$, notice that $f(x) = (g \circ h \circ j)(x)$. Thus we will be using the chain rule twice, thus $f' = g'(h \circ j)\,(h \circ j)' = g'(h \circ j)\, h'(j)\, j'$ or in other words

$$\begin{aligned}\frac{d}{dx}(f(x)) &= \frac{d}{dx}(\ln(\sin^2(x))) = \frac{1}{\sin^2(x)}\frac{d}{dx}(\sin^2(x)) = \frac{1}{\sin^2(x)}(2\sin(x))\frac{d}{dx}(\sin(x)) \\ &= \frac{2\sin(x)\cos(x)}{\sin^2(x)} = 2\frac{\cos(x)}{\sin(x)} = 2\cot(x)\,.\end{aligned}$$

1.3. **19.5 (e).** Use the power rule to find the first and second derivatives of the function $M(t) = \frac{2}{3}t^{3/2}$.
Solution: Notice the following

$$\begin{aligned}\frac{d}{dt}(M(t)) &= \frac{d}{dt}\left(\frac{2}{3}t^{3/2}\right) = \frac{2}{3}\frac{d}{dt}(t^{3/2}) = \frac{2}{3}\frac{3}{2}t^{\frac{3}{2}-1} = t^{1/2} \qquad \text{by power rule} \\ \frac{d^2}{dt^2}(M(t)) &= \frac{d}{dt}(M'(t)) = \frac{d}{dt}(t^{1/2}) = \frac{1}{2}t^{-1/2} = \frac{1}{2\sqrt{t}}\end{aligned}$$

1.4. **19.11.** Using the derivative rules, show that if $s(t) = A\cos(\omega t + \delta)$, where A, ω, and δ are constants, then $s''(t) = -\omega^2 s(t)$.
Solution: Given A, ω and δ constats and $s(t) = A\cos(\omega t + \delta)$ we notice that the first derivative can be obtained through chain rule

$$\frac{d}{dt}s(t) = \frac{d}{dt}\left(A\cos(\omega t + \delta)\right) = A\frac{d}{dt}\cos(\omega t + \delta) = A(-\sin(\omega t + \delta))\left(\frac{d}{dt}(\omega t + \delta)\right) = -A\omega\sin(\omega t + \delta)$$

We then observe that the second derivative can be obtained by applying chain rule once again

$$\frac{d^2}{dt^2}s(t) = \frac{d}{dt}\left(-A\omega\sin(\omega t + \delta)\right) = -A\omega\frac{d}{dt}(\sin(\omega t + \delta)) = -A\omega\cos(\omega t + \delta)\frac{d}{dt}(\omega t + \delta) = -A\omega^2\cos(\omega t + \delta) = -\omega^2 s(t)$$

Thus showing that $s''(t) = -\omega^2 s(t)$.

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1.5. **19.15.** A yam is put into a hot oven, maintained at a constant temperature of 200°C. At time $t = 30$ minutes, the temperature T of the yam is 120° and is increasing at an instantaneous rate of 2°/min. Newton's Law of Cooling (or, in our case, warming) implies that the temperature at time t is given by $T(t) = 200 - ae^{-bt}$. Find a and b.

Solution: From the problem statement we can extract the following information: $T(30) = 120°$ and $T'(30) = 2°/\text{min}$. Then by using the model provided by Newton's Law of Cooling and its derivative we can use this information in order to find the value for the constants a and b. Notice

$$T(t) = 200 - ae^{-bt} \qquad \text{and} \qquad T'(t) = abe^{-bt}$$

by evaluating both functions at $t = 30$ we have that

$$120 = T(30) = 200 - ae^{-30b}$$

and

$$2 = T'(30) = abe^{-30b}$$

Thus we can deduce that $ae^{-30b} = \frac{2}{b}$ by substituting this on the equation of $T(30)$ we obtain $\frac{2}{b} = 200 - 120 = 80$. Hence $b = \frac{1}{40}$. Then we can find the value of a by substituting on the equation concerning $T'(30)$. Thus having

$$80 = ae^{-30/40} \qquad \Rightarrow \qquad a = 80e^{3/4}$$

Thus having that $b = \frac{1}{40}$ and $a = 80e^{3/4}$.

1.6. **20.4.** Find the absolute maximum and minimum of these functions on $[-1, 4]$.

(a) $f(x) = x^2 - 1$.
(b) $f(x) = x^3 + x$.

Solution: We have the following

(a) Consider the function $f(x) = x^2 - 1$, whose derivative is $f'(x) = 2x$. By solving $f'(x) = 0$ we find the candidate for maximum or minimum, thus $x = 0$ is such candidate and since $f''(x) = 2$ we know that this is a local minimum. Moreover notice that $f(-1) = 0$, $f(0) = -1$ and $f(4) = 15$. Thus having that $x = 0$ is an absolute minimum and $x = 4$ is an absolute maximum.
(b) Consider the function $f(x) = x^3 + x$, whose derivative is $f'(x) = 3x^2 + 1$. We notice that the solutions to $f'(x) = 0$ are not on the real numbers, thus there is no candidate for local maximum or minimum. Moreover, since $f'(x) > 0$ in the interval $[-1, 4]$ by first derivative test we know that $f(x)$ is strictly increasing. Finally, since $f(-1) = -2$ and $f(4) = 68$, we know that the absolute minimum is at $x = -1$ and the absolute minimum is at $x = 4$.

1.7. **20.6.** The population of a flock of geese is modeled by

$$G(t) = 2000 + 150\sin\left(\frac{\pi}{6}t\right) + 75\sin\left(\frac{\pi}{3}t\right) .$$

where t is measured in months from January 1.

(a) Find any critical points and any inflection points of G. State the location of any local maxima, minima, and inflection points. Use this information to sketch a graph of $G(t)$ over a 2 year period.
(b) Interpret any local maxima, minima, and inflection points over the first 2 years in biological terms.
(c) How fast is the flock growing on May 1? On May 1, is the rate at which the flock is growing increasing or decreasing?

Solution: Let

$$G(t) = 2000 + 150\sin\left(\frac{\pi}{6}t\right) + 75\sin\left(\frac{\pi}{3}t\right) .$$

and notice that

$$G'(t) = 25\pi\left(\cos\left(\frac{\pi}{6}t\right) + \cos\left(\frac{\pi}{3}t\right)\right)$$

moreover we also notice that

$$G''(t) = -\frac{25\pi^2}{3}\left(\frac{1}{2}\sin\left(\frac{\pi}{6}t\right) + \sin\left(\frac{\pi}{3}t\right)\right)$$

(a) We first look for the candidates for maximum and minimum. Thus we solve $G'(t) = 0$, let us notice that this is equivalent to find the solutions to

$$\cos\left(\frac{\pi}{6}t\right) + \cos\left(\frac{\pi}{3}t\right) = 0$$

If we define $\theta = \frac{\pi}{6}t$, the previous equation is substituted by $\cos(\theta) + \cos(2\theta) = 0$. By using the trigonometric identity $\cos(2\theta) = 2\cos^2(\theta) - 1$. Thus needing to solve the quadratic equation

$$\cos(\theta) + 2\cos^2(\theta) - 1 = 0$$

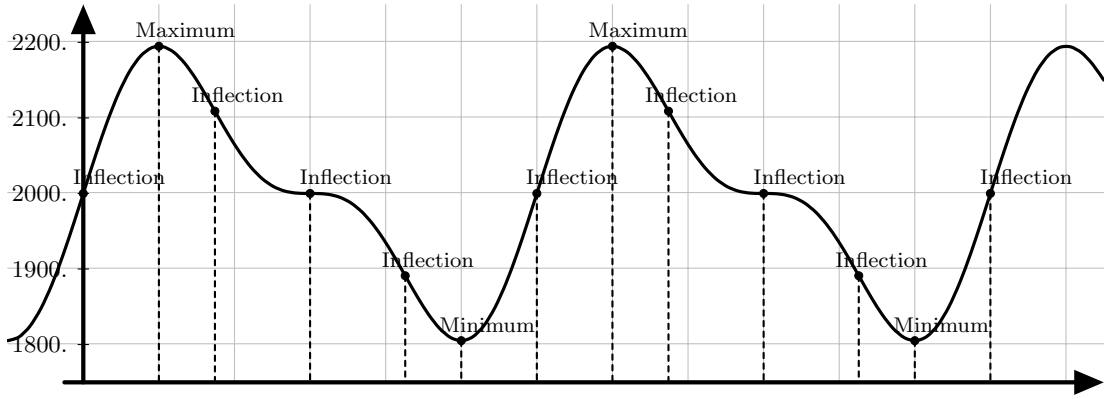
Thus having the solution $\cos(\theta) = -1$ and $\cos(\theta) = \frac{1}{2}$. From the solution $\cos(\theta) = -1$ we have that $\theta = \pi + 2n\pi$ with $n \in \mathbb{Z}$, since $\theta = \frac{\pi}{6}t$ we have that $t = 6 + 12n$. From the solution $\cos(\theta) = \frac{1}{2}$ we have that $\theta = 2n\pi + \frac{\pi}{3}$ and $\theta = 2n\pi - \frac{\pi}{3}$, since $theta = \frac{\pi}{6}t$ we have that $t = 2 + 12n$ and $t = 12n - 2$ for $n \in \mathbb{Z}$. We only restrict to the interval $[0, 24]$. Thus having candidates on $t = 2, 6, 10, 14, 18, 22$. By using the second derivative we notice that $t = 2, 14$ are local maximum and $t = 10, 22$ are local minimum.

Moreover since $G''(t) = 0$ when $\frac{1}{2}\sin\left(\frac{\pi}{6}t\right) + \sin\left(\frac{\pi}{3}t\right) = 0$. Once again we let $\theta = \frac{\pi}{6}t$, thus solving $\frac{1}{2}\sin(\theta) + \sin(2\theta) = 0$, since $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ we have to find the solution to

$$2\sin(\theta)\left(\frac{1}{4} + \cos(\theta)\right) = 0$$

Hence either $\sin(\theta) = 0$ or $\cos(\theta) = -\frac{1}{4}$. From $\sin(\theta) = 0$ we have that $\theta = n\pi$ thus we obtain that $t = 6n$ with $n \in \mathbb{Z}$. From $\cos(\theta) = -\frac{1}{4}$, we have that $\theta = 2n\pi + \arccos\left(-\frac{1}{4}\right)$ and $\theta = 2n\pi - \arccos\left(-\frac{1}{4}\right)$; hence $t = 12n - 6\arccos\left(-\frac{1}{4}\right)/\pi \approx 12n - 3.4826$ and $t = 12n + 6\arccos\left(-\frac{1}{4}\right)/\pi \approx 12n + 3.4826$. Thus having that the inflection points are at $t = 0, 6, 12, 18, 24, 3.4826, 15.4826, 8.5174, 20.5174$.

If we sketch the graph of this function we obtain the following.



(b) The maximum points on $G(t)$ indicate that the number of geese in the flock was the highest during that season. Similarly, the minimum indicate a low population of geese on the flock. Finally, the inflection points indicate a point of maximum population growth when $t = 0, 6, 12, 18, 22$ and minimum population growth at $t = 3.4826, 8.5174, 15.4826, 20.5174$.

(c) What we are asked to find is the second derivative of $G(t)$ evaluated at $t = 4$. The second derivative at a certain time will give us how fast is the flock of geese growing. Notice that $G''(4) \approx 35.6139$ since this second derivative is positive we know that the rate at which the flow is growing is increasing.

References

[1] Suzanne Bodine, Erin N.; Lenhart and Louis J. Gross. *Mathematics for the life sciences*. Princeton University Press, Princeton New Jersey, 2014.