Lecture notes MTH 228, Fall 2017

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December 2, 2017

Part I: Differentiation

Lect1, 9/20: Review of derivatives.

Suppose we are given a function $f(x)$, where x belongs to some interval (a, b), where $a < b$ and a and b could possibly be $-\infty$ and $+\infty$ respectively. Let x_0 belong to (a, b).

We say that f **is differentiable at** x_0 if the following limit exists:

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x)}{h}$$

If f is differentiable at x_0, we denote the limit above by

$$f'(x_0) \text{ or } \frac{df}{dx}(x_0),$$

and call it the **derivative of** f **at** x_0.

The geometrical interpretation of $f'(x_0)$ is that it represents the **slope of the tangent line** to the graph of f at the point x_0.

In applications, $f(x)$ could represent the density of a microbial population at a certain location x, and in this case $f'(x)$ represents the spatial gradient of that density. But if x represents "time" (as it often does, although we usually write $f(t)$ in that case), then $f'(x)$ represents the rate of change of the microbial density at time x.

Example 1: Suppose that n is a positive integer, and let

$$f(x) = x^n, \quad -\infty < x < +\infty$$

We claim that f is differentiable at any x, and that

$$f'(x) = nx^{n-1}$$

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To see this, we compute the above limit:

$$\begin{aligned}
\lim_{h\to 0}\frac{(x+h)^n - x^n}{h} &= \lim_{h\to 0}\frac{\left(\sum_{i=0}^{n}\binom{n}{i}x^{n-i}h^i\right) - x^n}{h} \\
&= \lim_{h\to 0}\frac{(x^n + nx^{n-1}h + n(n-1)x^{n-2}h^2/2 + \cdots + h^n) - x^n}{h} \\
&= nx^{n-1} + \lim_{h\to 0} p(h) \\
&= nx^{n-1} + 0,
\end{aligned}$$

because $p(h)$ is a polynomial of degree $n-1$ in h with coefficients that depend on x and n, and which is missing a constant term (i.e. the term corresponding to h^0).

In the first line, we used an important expansion for the nth power of the sum of x and h:

$$(x+h)^n = \sum_{i=0}^{n}\binom{n}{i}x^{n-i}h^i,$$

called the **binomial formula**. Recall that $\binom{n}{i}$ means:

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}, \text{ and } n! = n(n-1)\ldots 2.1 \text{ (and by convention also } 0! = 1)$$

Example 2: Let

$$f(x) = |x|, \quad \infty < x < +\infty$$

Plot the graph of this function, and think about the geometrical interpretation of the derivative. We will show that f is differentiable at any $x \neq 0$, but **not** differentiable at $x = 0$.

To show that f is not differentiable at $x = 0$, we claim that the following limit does not exist:

$$\lim_{h\to 0}\frac{|0+h| - 0}{h} = \lim_{h\to 0}\frac{|h|}{h}$$

Indeed, if $h > 0$, then $|h| = h$ and the above limit is a right-limit which equals $+1$; if $h < 0$, then $|h| = -h$ and the above limit is a left-limit which equals -1. Since $+1$ and -1 are unequal, the claim is proved.

If $x \neq 0$, then f is differentiable with derivative:

$$f'(x) = \begin{cases} -1, & \text{if } x < 0 \\ +1, & \text{if } x > 0 \end{cases}$$

To see this, let's consider the case that $x < 0$. Then for all sufficiently small h (both negative *and* positive!), $x + h$ will be negative as well. But then $|x+h| = -(x+h)$, and thus the limit:

$$\lim_{h\to 0}\frac{|x+h| - |x|}{h} = \lim_{h\to 0}\frac{-(x+h) - (-x)}{h} = \lim_{h\to 0}\frac{-h}{h} = -1$$

Can you show that $f'(x) = +1$ when $x > 0$?

Lect2, 9/22: Review of properties of derivatives.

We now state the derivatives of several well-known and important functions, and also review various properties of derivatives.

Derivatives of important functions

1. **Exponential function**: For all $-\infty < x < +\infty$: $\frac{d}{dx}\,\mathrm{e}^x = \mathrm{e}^x$.
2. **Natural logarithm**: For all $x > 0$: $\frac{d}{dx}\ln x = \frac{1}{x}$.
3. **Elementary trig functions**: For all $-\infty < x < +\infty$: $\frac{d}{dx}\sin x = \cos x$ and $\frac{d}{dx}\cos x = -\sin x$.
4. **Powers I**: Let b be given. For all $x > 0$: $\frac{d}{dx}x^b = bx^{b-1}$.
5. **Powers II**: Let $a > 0$ be given. For all $-\infty < x < +\infty$: $\frac{d}{dx}a^x = \ln a\, a^x$.

Proving the validity of these formulas, starting from the definition of the derivative reviewed in Lect 1, is not that easy. In case of the exponential and elementary trig functions, these proofs are given in the text in Appendix B, p. 574-576. Fortunately, the other formulas can be much more easily verified using the following properties of derivatives. We shall illustrate this below.

Properties of derivatives Assume that $f(x)$ and $g(x)$ are given functions, each defined on some open interval. Both functions are assumed to be differentiable at any point of their domain.

1. **Derivative of a sum (The Sum Rule)**: $(f(x) + g(x))' = f'(x) + g'(x)$
2. **Derivative of a product (The Product Rule)**: $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$
3. **Derivative of a quotient (The Quotient Rule)**: If $g(x) \neq 0$, then $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g'(x)}{g^2(x)}$
4. **Derivative of a composition (The Chain Rule)**: $(f(g(x)))' = f'(g(x))g'(x)$

Examples: 1. For any $x > 0$ we know that $\mathrm{e}^{\ln x} = x$ because the exponential function and natural logarithm are inverse functions to each other. We can write this as a composition:

$$f(g(x)) = x, \text{ for all } x > 0, \text{ with } f(x) = \mathrm{e}^x \text{ and } g(x) = \ln x$$

Taking derivatives, using the product rule, the fact that $d/dx\,\mathrm{e}^x = \mathrm{e}^x$, and the fact shown in the last lecture that $d/dx(x) = 1$, we get:

$$\mathrm{e}^{\ln x}\frac{d}{dx}\ln x = 1, \text{ for all } x > 0,$$

from which follows that

$$\frac{d}{dx}\ln x = \frac{1}{x},$$

showing that the derivative of the natural logarithm is as we expected.

2. Recall that if $x > 0$ and b is a given number, then

$$x^b = \mathrm{e}^{b \ln x}.$$

Setting $f(x) = \mathrm{e}^x$ and $g(x) = b \ln x$, this can be rewritten as as composition:

$$x^b = f(g(x))$$

Taking derivatives, and using the Chain Rule, and then the Product Rule and the result from Example 1 above, yields:

$$\frac{d}{dx} x^b = \mathrm{e}^{b \ln x} \frac{d}{dx} (b \ln x) = x^b \left(0 + b \frac{1}{x} \right) = b x^{b-1},$$

also as expected.

3. (Inspired by some of my own research on the Tragedy of the Commons; google this last term.) Let $x_1(t)$ and $x_2(t)$ be the positive densities at time t of two microbial populations in a bioreactor, called the *cooperator* and *cheater* respectively. The rate of change of these densities is given by the following equations:

$$\begin{aligned} \frac{d}{dt} x_1(t) &= x_1(t)(q f(p(t)) - 1) \\ \frac{d}{dt} x_2(t) &= x_2(t)(f(p(t)) - 1) \end{aligned}$$

Here $p(t)$ denotes the (continuous and positive) concentration at time t of the nutrient which both microbial populations are consuming for their own growth. The function $f(p)$ is known as the per capita growth rate function of the cheater. This function is assumed to be continuous and positive for $p > 0$. The parameter q is assumed to belong to the interval $(0, 1)$. Each of these two rate of change equations consists of the difference of two terms. The first term captures the growth rate, the second represents the rate at which the microbes flow out of the bioreactor. Note in particular that the per capita growth rate function of the cooperator is equal to a fraction q of the per capita growth rate function of the cheater. That is, the per capita growth rate of the cooperator is always smaller than that of the cheater.

We claim that the rate of change of the quotient of the densities of cooperator and cheater, is always negative. Consequently, this ratio of densities is always decreasing.

To show this, we apply the Quotient Rule to the quotient $x_1(t)/x_2(t)$, and use the rate of change equations for cooperator and cheater:

$$\begin{aligned} \frac{d}{dt} \left(\frac{x_1(t)}{x_2(t)} \right) &= \frac{x_1'(t) x_2(t) - x_1(t) x_2'(t)}{x_2^2(t)} \\ &= \frac{x_1(t)(q f(p(t)) - 1) x_2(t) - x_1(t) x_2(t)(f(p(t)) - 1)}{x_2^2(t)} \\ &= (q-1) \frac{x_1(t) x_2(t) f(p(t))}{x_2^2(t)} \\ &= (q-1) \frac{x_1(t) f(p(t))}{x_2(t)} < 0, \text{ for all time } t \end{aligned}$$

because $x_1(t)$, $x_2(t)$ and $f(p(t))$ are positive functions, and $q < 1$.

Lect3, 9/25: Finding minima and maxima.

Sample problem: Dr Wannabee is growing bacteria in a Petri dish and has set up a nutrient supply system in such a way that the density of the bacteria over a period of 24 hours is modeled by:

$$f(t) = 100 + t\sin(\pi t/12),\ 0 \leq t \leq 24h$$

When does the density of the bacteria peak, and what is their peak density?

Let f be a given function with domain D. We say that f has a **local maximum** at x_0 in D if $f(x) \leq f(x_0)$, for all x belonging to D and to some open interval containing x_0. Similarly we say that f has a **local minimum** at x_0 in D if $f(x) \geq f(x_0)$, for all x belonging to D and to some open interval containing x_0. We say that f has a **global maximum** at x_0 in D if $f(x) \leq f(x_0)$, for all x belonging to D. Can you define a **global minimum**?

An important result from Calculus, known as the **extreme value theorem**, is that if f is a *continuous* function, defined on a *compact interval* $[a, b]$ with $a < b$, then f has a global maximum and a global minimum.

Examples: Suppose that $f(x) = x$ with x in $D = [0, 1]$. Then clearly f has a local and global minimum at $x = 0$ and a local and global maximum at $x = 1$. Suppose we consider the same function, but take its domain to be $D = (0, 1)$. In this case, there is no local or global minimum, and no local or global maximum. The extreme value theorem does not apply in this case because this domain is not a compact interval. The same conclusion would hold if we consider the same function, but now with domain $D = (-\infty, +\infty)$, which also fails to be compact.

Suppose that $g(x) = 1$ with x in $D = [0, 1]$. Then **every** x in D is local and global maximum, as well as a local and global maximum. The latter example shows that global maxima and global minima are not necessarily unique.

Q: **How do we find local and global minima and maxima, provided they exist?**

To answer this question, derivatives turn out to be quite useful. From now on, we assume that *all the functions considered are differentiable* at every interior point of their domain (a point is an interior point of the domain, if it is contained in an open interval which in turn is contained in the domain).and that the derivative $f'(x)$ is continuous there. We claim that:

First Derivative Test: *Suppose that f has a local maximum or a local minimum at an interior point x_0 of its domain D. Then $f'(x_0) = 0$.*

Indeed, if $f'(x_0)$ was positive or negative, then we can always find a point x near x_0 where $f(x) > f(x_0)$, and another point y near x_0 where $f(x) < f(x_0)$, which implies that x_0 cannot be a local minimum or a local maximum.

The main idea expressed above is that the graph of a differentiable function which achieves a local maximum or minimum at an interior point of its domain, must be flat there. Let us immediately emphasize that the converse is not true. For example, the function $f(x) = x^3$, with x in $[-1, +1]$, has zero derivative at $x_0 = 0$, but does not achieve a local minimum or local maximum there (inspect the graph).

Nevertheless, the First Derivative Test is useful because it identifies a usually small number of points which are candidates for local minima and local maxima.

Notice that the First Derivative Test only applies to interior points of the domain. What about points that are not interior points? This is a bit trickier and best clarified with a simple example. Suppose that $f(x) = x$ with domain $D = [0, 1]$. We have already seen above that f has a local and global minimum at $x_0 = 0$. But $x_0 = 0$ is not an interior point of the domain of f, and $f'(0) = +1$ which is not zero. In fact, the function is increasing near $x_0 = 0$ (because $f'(0) > 1$), and the domain of the function only extends to the "right" of $x_0 = 0$. Therefore, $x_0 = 0$ is a local minimum despite the fact that the derivative is not zero there.

To determine whether points that don't belong to the interior of the domain are local minima or maxima, we typically have do a separate analysis as we just did for the simple example. Usually there are very few of those points anyway (exactly 2 in case the domain is a compact interval for instance).

Q: **I have found all interior points of the domain where the derivative is zero. How do I decide if I have a local maximum or minimum?**

To answer this question, we impose a stronger condition, namely that f has a continuous second derivative in all points that are interior points of its domain. This means that the following limit, which represents the derivative of the derivative (hence "2nd derivative") and will be denoted as $f''(x_0)$, exists:

$$\lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

at all points x which are interior points of the domain of f.

We now claim that:

Second Derivative Test: *Suppose that $f'(x_0) = 0$ at an interior point x_0 of the domain D of f. If $f''(x_0) < 0$ then f has a local maximum at x_0. If $f''(x_0) > 0$, then f has a local minimum at x_0.*

To get some intuition about the meaning of this, suppose that we have an interior point with $f'(x_0) = 0$ and $f''(x_0) < 0$. The last inequality means that at x_0, the derivative $f'(x)$ is *decreasing*. In other words, the slope of the tangent line to the graph, is getting smaller near x_0. Since this slope is zero at x_0, this means that for x to the left of x_0, this slope is positive, and for x to the right, this slope is negative. But then x_0 must be a local maximum.

Lect4, 9/27: Examples of finding minima and maxima.

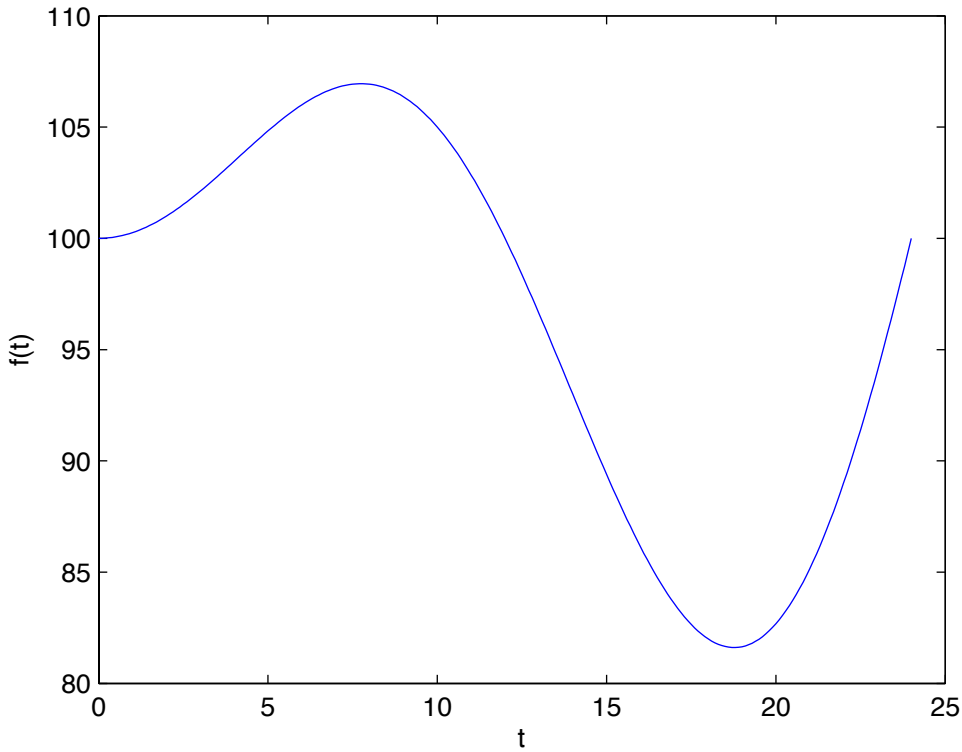


Figure 1: Bacterial density versus time of Dr Wannabee's experiment.

Sample problem: Before we start, we plot the function $f(t)$, to get an idea of what to expect, see Figure 1. The derivative of the function $f(t) = 100 + t\sin(\pi t/12)$, with $0 \le t \le 24h$ is:

$$f'(t) = \sin(\pi t/12) + \frac{\pi t}{12}\cos(\pi t/12)$$

We wish to apply the First Derivative Test, and set $f'(t) = 0$. This is equivalent to

$$\tan(\pi t/12) = -\frac{\pi t}{12},\ 0 \le t \le 24.$$

It is clear from graphing the two functions on the left and the right (Do this!), that there are 3 points in the domain where their graphs intersect: $t_0 = 0$, t_1 and t_2. We can find t_1 and t_2 numerically,with a calculator, or using software such as Matlab:

$$t_1 = 7.7493,\ \text{and}\ t_2 = 18.7670.$$

Only t_1 and t_2 are interior points of the domain $[0, 24]$. To see if $f(t)$ has a local minimum or local maximum at t_1 and at t_2, we find the second derivative:

$$f''(t) = \frac{\pi}{12}\left(2\cos(\pi t/12) - \frac{\pi t}{12}\sin(\pi t/12)\right),$$

and evaluate it numerically at t_1 and t_2:

$$f''(t_1) = -0.7079 < 0, \text{ and } f''(t_2) = 1.3649 > 0,$$

and then the Second Derivative Test implies that $f(t)$ has a local maximum at $t = t_1$ and a local minimum at $t = t_2$. We now consider the 2 points of the domain that are not interior points: $t = 0$ and $t = 24$. We first investigate the function near $t = 0$. Evaluating $f(0)$ and $f'(0)$ yields:

$$f(0) = f'(0) = 0,$$

so this is not enough information to conclude if f has a local minimum or maximum near $t = 0$. However, considering the definition of $f(t)$ directly, we se that for small but positive t, $f(t)$ will be larger than 100, and thus f has a local minimum at $t = 0$. Near $t = 24$, there holds:

$$f(24) = 100, \text{ and } f'(24) = 2\pi > 0,$$

and thus f has a local maximum at $t = 24$.

Summary: $f(t)$ has 2 local minima (at $t = 0$ and $t = t_2$), and two local maxima (at $t = t_1$ and $t = 24$) on its domain $[0, 24]$. Since $f(24) < f(t_1)$, we conclude that $f(t)$ has a unique global maximum at $t = t_1$. The density of Dr Wannabee's bacterial population therefore peaks at $t = t_1 = 7.7493h$ into the experiment, and reaches density of $f(t_1) = 106.95$ at that time.

Example 2: Now we consider a simpler example which can be solved with pencil and paper only. Suppose that

$$f(x) = \frac{x^3}{3} - \frac{3}{2}x^2 + 2x + 1, \ 0 \leq x \leq 3.$$

Find the local and global minima and maxima of f.

Let's compute the first and second derivatives:

$$f'(x) = x^2 - 3x + 2 = (x - 1)(x - 2), \text{ and } f''(x) = 2x - 3,$$

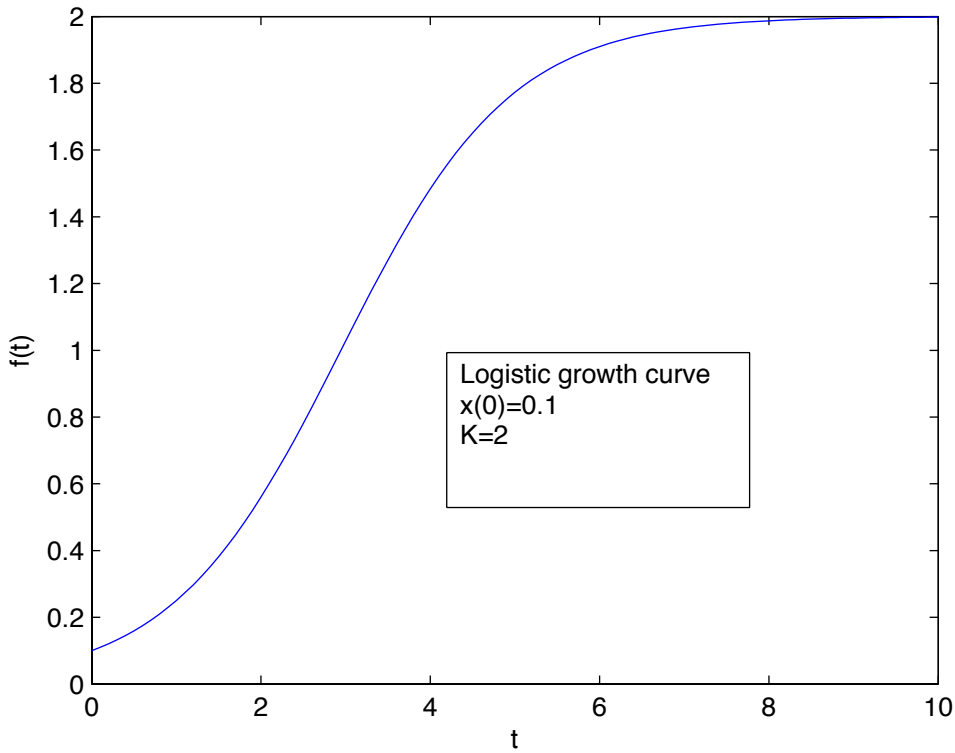
Then $f'(x) = 0$ at $x = 1$ and $x = 2$ which are both interior points of the domain. Moreover, $f''(1) = -1 < 0$ and $f''(2) = 1 > 0$, and therefore f has a local maximum at $x = 1$, and a local minimum at $x = 2$. The value of f at these points is $f(1) = 11/6$ and $f(2) = -1/3$ respectively. We now investigate f at the points $x = 0$ and $x = 3$. There holds that $f'(0) = f'(3) = 2 > 0$, and thus f is increasing near these two points. Consequently, f has a local minimum at $x = 0$, and a local maximum at $x = 3$. The value of f at these points is given by $f(0) = 1$ and $f(3) = 5/2$. Since $f(1) = 11/6 < 5/2 = f(3)$, f has a unique global maximum at $x = 3$, and a unique global minimum at $x = 2$.

lect5, 6 and 7, 9/29, 10/2 and 10/4

Example 3: Dr Wannabee did another experiment in a Petri dish, growing *E. coli* on glucose (sugar) and observed that the time series data of the density could be fitted with a particularly elegant formula:

$$f(t) = \frac{1}{\frac{1}{x(0)}\,\mathrm{e}^{-t} + \frac{1}{K}\left(1 - \mathrm{e}^{-t}\right)},\ t \geq 0,$$

and where $x(0) > 0$ represents the initial bacterial density at the start of the experiment, and $K > 0$ is the density Dr W observed after a long time had passed. In the figure above, we plot a typical time series of this curve, known as a *logistic growth curve*.



1. Show that if $x(0) < K$, the function $f(t)$ is always increasing by showing that $f'(t) > 0$ for all $t \geq 0$.

2. Show that if $2x(0) < K$, the function $f(t)$ has a unique inflection point.[1]

[1] A function $f(t)$ is said to have an *inflection point* at time t^* if the second derivative of $f(t)$ is zero at t^*, and switches sign there; i.e. the sign of $f'(t)$ is different for every pair of times that are close to, but on opposite sides of t^*.

1. The first derivative of $f(t)$ is:

$$\begin{aligned}
f'(t) &= \left(-\frac{1}{x(0)}\,\mathrm{e}^{-t}+\frac{1}{K}\,\mathrm{e}^{-t}\right)\cdot\frac{-1}{\left(\frac{1}{x(0)}\,\mathrm{e}^{-t}+\frac{1}{K}\,(1-\mathrm{e}^{-t})\right)^2}\\
&= \left(\frac{1}{x(0)}-\frac{1}{K}\right)\frac{\mathrm{e}^{-t}}{\left(\frac{1}{x(0)}\,\mathrm{e}^{-t}+\frac{1}{K}\,(1-\mathrm{e}^{-t})\right)^2}\\
&= \left(\frac{1}{x(0)}-\frac{1}{K}\right)\frac{1}{\left(\mathrm{e}^{t/2}\left(\frac{1}{x(0)}\,\mathrm{e}^{-t}+\frac{1}{K}\,(1-\mathrm{e}^{-t})\right)\right)^2}\\
&= \left(\frac{1}{x(0)}-\frac{1}{K}\right)\frac{1}{\left(\frac{1}{K}\,\mathrm{e}^{t/2}+\left(\frac{1}{x(0)}-\frac{1}{K}\right)\mathrm{e}^{-t/2}\right)^2}
\end{aligned}$$

The second line above already clearly shows that $f'(t) > 0$ for all $t \geq 0$, whenever $x(0) < K$, establishing that the bacteria are always on the rise. We manipulated the expression in the subsequent lines to facilitate the calculation of the second derivative of $f(t)$ in the next item.

2. The second derivative of $f(t)$ is obtained by calculating the derivative of the last expression above:

$$f''(t) = \left(\frac{1}{x(0)}-\frac{1}{K}\right)\left(\frac{1}{K}\frac{\mathrm{e}^{t/2}}{2}-\left(\frac{1}{x(0)}-\frac{1}{K}\right)\frac{\mathrm{e}^{-t/2}}{2}\right)\frac{-2}{\left(\frac{1}{K}\,\mathrm{e}^{t/2}+\left(\frac{1}{x(0)}-\frac{1}{K}\right)\mathrm{e}^{-t/2}\right)^3}$$

We note that $f''(t)$ equals the product of 3 factors, and that the first factor is positive because $x(0) < K$, and similarly that the third factor is negative for all $t \geq 0$, again because $x(0) < K$.

Thus, we investigate the sign of the middle factor. Is there a time t^* where the middle factor is zero? Setting it equal to zero, leads to the equation:

$$\frac{1}{K}\frac{\mathrm{e}^{t/2}}{2} = \left(\frac{1}{x(0)}-\frac{1}{K}\right)\frac{\mathrm{e}^{-t/2}}{2}, \text{ or after simplifying a bit:}$$

$$\mathrm{e}^t = \frac{K}{x(0)} - 1.$$

Again using that $x(0) < K$, we can solve the latter equation for t, and we find that there is a unique t^*, given by the formula

$$t^* = \ln\left(\frac{K}{x(0)} - 1\right),$$

having the property that $f''(t^*) = 0$. Notice that $t^* > 0$ because we assumed that $2x(0) < K$.

Finally, if $t < t^*$, then taking exponentials on both sides (and recalling that the exponential function is increasing):

$$\mathrm{e}^t < \frac{K}{x(0)} - 1, \text{ and therefore also}$$

$$\frac{1}{K}\frac{e^{t/2}}{2} < \left(\frac{1}{x(0)} - \frac{1}{K}\right)\frac{e^{-t/2}}{2}.$$

Similarly,we can show that if $t > t^*$, then

$$\frac{1}{K}\frac{e^{t/2}}{2} > \left(\frac{1}{x(0)} - \frac{1}{K}\right)\frac{e^{-t/2}}{2}.$$

Thus, for a pair of times t belonging to opposite sides of t^*, the middle factor in $f''(t)$ has opposite signs, but that implies that $f''(t)$ has opposite signs at these times as well. In other words, we have shown that $f(t)$ exhibits an inflection point at t^*. More precisely, $f'(t)$ is positive if $t < t^*$, zero if $t = t^*$, and negative if $t > t^*$.

Summary: What do these results tell us about the growth curve of the bacteria? First, note that at time $t = 0$, the density equals $x(0)$ and that as $t \to \infty$, the density approaches K because:

$$\lim_{t\to\infty} x(t) = K.$$

If $x(0) < K$, then item 1 tells us that the density is an increasing function of time. If $2x(0) < K$, then item 2 tells us that at time t^*, the derivative of $f'(t)$ switches sign, going from positive to negative. This means that the rate of change of the slope of the tangent line, switches from being positive to negative. In other words, the growth of the bacteria accelerates at early times, but slows down in later times.

This is an important feature of what is known as the *logistic growth* of the bacteria, and it is frequently observed in experiments. Final comment about terminology: In applications one usually refers to the parameter K as the *carrying capacity*, reflecting that this is the value at which the density settles after a long time (recall that $\lim_{t\to+\infty} f(t) = K$).

Example 4: Optimal clutch size (this is a variation of Example 20.14 in the text)

In many physical and biological systems, it is observed that the state of the system adopts values which either minimize or maximize certain functions. For example, in protein folding, the state of the folded protein corresponds to a configuration which minimizes the energy of the protein. In biological applications, states are often believed to adopt a value which *maximizes a fitness function*. We shall consider an example of this type here, where the problem is to determine the optimal size of a clutch (the number of eggs laid by a bird) which maximizes the fitness of the clutch. Before turning to this problem, I'd like to emphasize that even defining a *fitness function* in a biological context is a notoriously difficult problem, and unfortunately, it is frequently *not defined*, making a proper analysis impossible. You, as a future Life Scientist should be very aware of this issue, and always make sure that fitness is defined unambiguously before attempting to understand the behavior of the system you're studying.

Consider a single bird that produces a clutch of size N. The clutch fitness is a function of the clutch size N, and it measures the potential for reproductive success of the entire collection of offspring. One reasonable choice for clutch fitness is:

$$\begin{aligned}\text{clutch fitness} &= \text{clutch size} \times \text{probability of survival of a single egg}\\ &= Ns(N)\end{aligned}$$

where the function $s(N)$ represents the probability of survival of a single egg in a clutch of size N. Note that we assume that this probability depends on the actual clutch size, rather than being a constant. One possible explanation for this, is that larger clutch sizes make it more difficult for individuals to acquire enough resources to survive (competition for limited resources). The fact that $s(N)$ represents a probability, implies that it must take values between 0 and 1. Let's make a particularly simple choice:

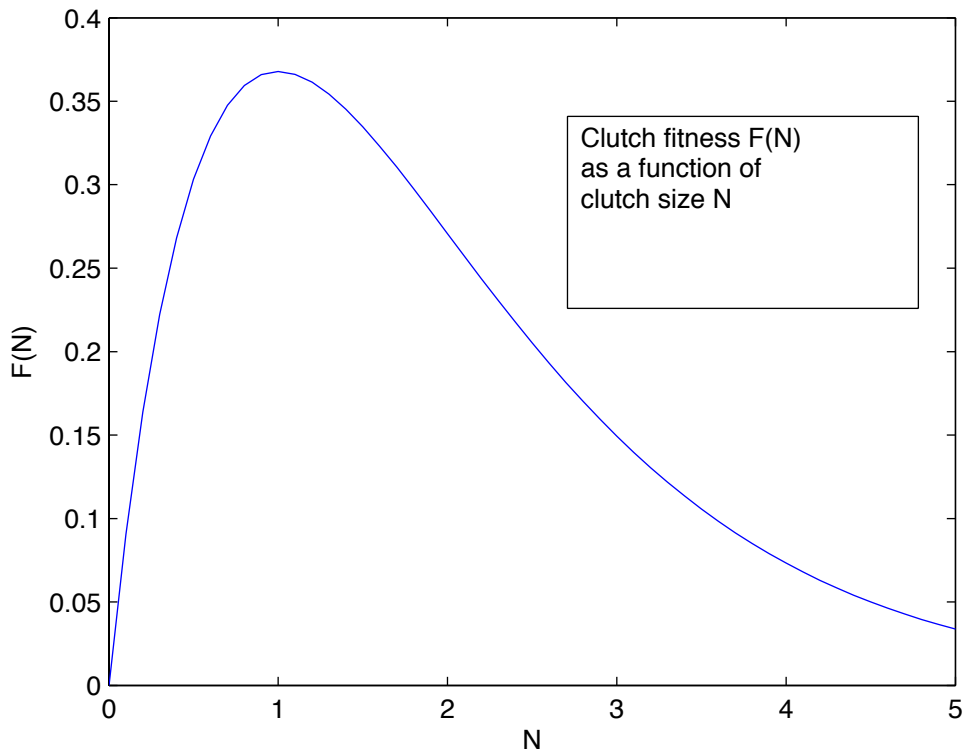
$$s(N) = \mathrm{e}^{-N}, \ N \geq 0.$$

Clearly, the function $s(N)$ takes values between 0 and 1, and it is decreasing (because $s'(N) = -\mathrm{e}^{-N} < 0$) from 1 (when $N = 0$), to 0 (since $\lim_{N\to\infty} s(N) = 0$).

We can now rephrase the optimal clutch size problem, as a precise mathematical problem, namely: Maximize the following function:

$$F(N) = N\,\mathrm{e}^{-N}, \ N \geq 0.$$

The graph of $F(N)$ is illustrated below.



We apply the First Derivative Test:

$$F'(N) = \mathrm{e}^{-N} - N\,\mathrm{e}^{-N} = (1 - N)\,\mathrm{e}^{-N} = 0,$$

and solving for N yields that N must equal N^*, where

$$N^* = 1.$$

The second derivative of $F(N)$ is:

$$F''(N) = -\,\mathrm{e}^{-N} - (1-N)\,\mathrm{e}^{-N} = (-2+N)\,\mathrm{e}^{-N},$$

and applying the Second Derivative Test, we see that:

$$F''(N^*) = -\,\mathrm{e}^{-1} < 0,$$

implying that $F(N)$ has a local maximum at $N^* = 1$. Finally, since $F(N)$ is increasing on $[0,1]$ (because $F'(N) > 0$ for N in $[0,1]$), and decreasing for $N > 1$ (because $F'(N) < 0$ for $N > 1$), the function $F(N)$ achieves a global maximum at $N = N^*$. We conclude that the optimal clutch size for this example is N^* =1.

We remark that other choices for $s(N)$, may yield very different values of the optimal clutch size.

lect 8, 10/6: Review of derivatives. Solved homework problem 20.14 and part of 20.12.

Part II: Integration

lect 9, 10/9

Sample questions:

1. To gain insight into the migration patterns of sharks, Dr Fishburne tags a great white shark with a sensor that transmits the shark's speed $v(t)$, measured in mi/h, over a period of 24hours (The time t is measured in hours.) Here's what the sensor recorded:

$$v(t) = \begin{cases} 2,\ 0 \leq t \leq 6 \\ 2 + \frac{1}{2}(t-6)^2,\ 6 < t \leq 12 \\ 2 + \frac{1}{2}(t-18)^2,\ 12 < t \leq 18 \\ 2,\ 18 < t \leq 24 \end{cases}$$

   What is the total distance covered by the shark during this day? What is its average speed?

2. Ms Smith is putting herself through college and is very happy because she has landed a job at Starbucks to help pay for tuition. She wants to get an idea of how busy she will be taking orders at the counter. Later in this course we will learn more about *continuous random variables*. An important example of such a random variable, is a so-called *exponentially distributed random variable*. For instance, the waiting time before a new customer shows up at the Starbucks counter can be modeled by an exponentially distributed random variable.

   What is the probability that Ms Smith must wait 2 minutes or less for the next customer to arrive (and is exhausted by the end of the day)? What is the probability that she has to wait more than 30 minutes (and is bored out of her mind)?

To solve these 2 real-life examples will require the **integration of certain functions**. Traditionally, integration is motivated by another mathematical problem: To determine the area enclosed between the horizontal axis and the graph of a given function. This is also how we approach the topic of integration, and we will return to the above two problems later.

Question: Given a continuous, non-negative function $f(x)$, defined for $a \leq x \leq b$, where $a < b$ are finite numbers. What is the area enclosed between the horizontal axis and the graph of f?

Let us start with some simple examples:

1. Suppose that $f(x) = c$ for all x, where c is some positive constant. In this case, the graph of f is a horizontal line segment, and the area will be given by the area of a rectangle whose base has length $b - a$, and whose height is c. Thus, we have that:

$$\text{area } = c(b-a).$$

2. Suppose that $f(x) = kx$ for all x, where k is some positive constant. Let's also take $a = 0$ and b is some positive constant. In this case, the graph of f is

a straight line through the origin with a positive slope k, and thus the area is the area of a triangle with a base of length b, and height of length $f(b) = kb$, yielding:

$$\text{area } = \frac{1}{2}b(kb) = k\frac{b^2}{2}.$$

3. Suppose that $f(x) = x^2$, with $0 \leq x \leq 1$. Since the graph of this function is not straight, but curved, it is no longer obvious how to calculate the area. We've hit a roadblock and must find another strategy to find the area!

We outline the 2 main ideas to deal with this problem:

1. The **first main idea to find the area** for a general continuous and positive function $f(x)$ with $a \leq x \leq b$, is to **approximate it**. One natural way to approximate the area is to partition the interval $[a, b]$ in n subintervals $[a_i, b_i]$, with $a_i < b_i$, for all $i = 1, \ldots, n$, in such a way that $a_1 = a$, $b_n = b$, and $a_{i+1} = b_i$ for all $i = 1, \ldots, n-1$. We can now *approximate the area*, by the sum of the areas of the rectangles that are based on the subintervals $[a_i, b_i]$ and have height $f(b_i)$ (Draw a picture!):

$$\text{approximate area } = \sum_{i=1}^{n} f(b_i)(b_i - a_i)$$

2. The **second main idea to find the area** is that we can increase the number n of subintervals that partition the interval $[a, b]$, and take a limit with n approaching $+\infty$, while assuming that the largest length of the subintervals decreases to zero:

$$\text{area } = \lim_{n \to +\infty} \sum_{i=1}^{n} f(b_i)(b_i - a_1)$$

This last step is a huge one: there is no obvious reason why this limit exists. Moreover, it is conceivable that different choices for sequences of the partitioning subintervals, could yield different values for the corresponding limits. It is a remarkable result from Calculus, that this limit is indeed well-defined, and independent of the choice of the sequence of the partitioning subintervals! [2]

[2] Even more is true: If we would replace $f(b_i)$ in the above limit by $f(c_i)$, where c_i is any number in the subinterval $[a_i, b_i]$, then the limit still exists and has the same value as the above limit.

lect10 10/11

Next we illustrate these 2 steps for our Roadblock example $f(x) = x^2$, $0 \leq x \leq 1$:

1. **Approximation**: The simplest case occurs when we pick $n = 1$. Then $a_1 = 0$ and $b_1 = 1$, there is only 1 rectangle and we find that:

$$\begin{aligned} \text{approximate area} &= \sum_{i=1}^{1} f(b_i)(b_i - a_i) = f(b_1)(b_1 - a_1) \\ &= f(1)(1-0) \\ &= 1^2(1-0) \\ &= 1. \end{aligned}$$

   Now consider the case when $n = 2$. Then we could pick $a_1 = 0$, $b_1 = a_2 = 1/2$ and $b_2 = 1$. There are 2 rectangles, and their areas add up to

$$\begin{aligned} \text{approximate area} &= \sum_{i=1}^{2} f(b_i)(b_i - a_i) = f(b_1)(b_1 - a_1) + f(b_2)(b_2 - a_2) \\ &= f(1/2)(1/2 - 0) + f(1)(1 - 1/2) \\ &= (1/2)^2(1/2) + (1)^2(1 - 1/2) \\ &= 1/8 + 1/2 \\ &= 5/8 \end{aligned}$$

   We could continue this process by considering the cases $n = 3, 4, \ldots$. Can you calculate the $n = 3$ case?

2. Assume that n is an arbitrary positive integer, and partition the interval $[0, 1]$ in subintervals of equal lengths $1/n$. This happens when we choose $a_i = (i-1)/n$ and $b_i = i/n$ for all $i = 1, \ldots, n$. We can now calculate the area:

$$\begin{aligned} \text{area} &= \lim_{n \to +\infty} \sum_{i=1}^{n} f(b_i)(b_i - a_i) \\ &= \lim_{n \to +\infty} \sum_{i=1}^{n} (i/n)^2 (i/n - (i-1)/n) \\ &= \lim_{n \to +\infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2 \end{aligned}$$

   The sum in the last line above is the sum of the squares of the first n positive integers, and fortunately there exists a nice closed form formula for it:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Plugging this back in, gives:

$$\begin{aligned}
\text{area} &= \lim_{n\to+\infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2 \\
&= \lim_{n\to+\infty} \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6} \\
&= \frac{2}{6} \\
&= \frac{1}{3}
\end{aligned}$$

We have just calculated the area enclosed between the horizontal axis and the graph of the function $f(x) = x^2$, when x ranges from 0 to 1. This area equals 1/3.

lect11 10/13

We have just learned a specific process to calculate the area between the horizontal axis and the graph of a positive, continuous function $f(x)$, where $a \leq x \leq b$. We have seen that this leads to the computation of a limit. Actually, when we examine the process, we realize that it can also be carried out for functions which are not necessarily positive on their domain. If we consider a continuous function whose graph is sometimes above, and sometimes below the horizontal axis, and if the corresponding limit exists, we find that its value equals the difference of the areas above, and those below the horizontal axis. That is, the areas of the parts where the graph is below the horizontal axis is counted negatively.

From now on, when we are given a continuous function $f(x)$, defined for $a \leq x \leq b$, and the above limit exists, we will use a **new notation** for its value:

$$\int_a^b f(x)dx$$

We refer to this as the **integral from a to b of the function f**. In this expression we call the function $f(x)$ **the integrand**, we call a and b the **integration bounds**, and we call x the **integration variable**. Using the letter x is quite arbitrary; we may choose any other letter, without changing the value of the integral. That is, the above integral could also be written as $\int_a^b f(t)dt$ or as $\int_a^b f(y)dy$.

We now state some **properties of integrals** which follow from the definition of the integral as a limit: Suppose that $f(x)$ and $g(x)$ continuous for $a \leq x \leq c$, that $a < b < c$ and k are given constants. Then:

1. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$.
2. $\int_a^b f(x) + g(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.
3. $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$.
4. $\int_b^a f(x)dx = -\int_a^b f(x)dx$.

Of course, depending on the function $f(x)$, the calculation of the integral (that is, of the limit) may be very difficult, which begs the following question:

Is there another, simpler way to find the integral?

We will see that frequently this is indeed possible, after we introduce a new concept, known as an **antiderivative**. This new notion will also put us in a position to state one of the most remarkable result from calculus, the **Fundamental Theorem of Calculus**.

Antiderivatives Suppose we are given a function $f(x)$ where x belongs to an open, possibly infinite, interval. We say that another function $F(x)$ **is an antiderivative of** $f(x)$ if $F(x)$ is differentiable in the interval, and moreover:

$$F'(x) = f(x).$$

In words, the derivative of an antiderivative F is equal to the given function f.

A first remark is that antiderivatives are never uniquely defined functions. Indeed, suppose that F is an antiderivative of f. Then the function $F(x)+c$, where c is an arbitrary constant, is also an antiderivative of f. Indeed, $(F(x)+c)' = F'(x) = f(x)$, where we used the Sum Rule of derivatives, and the fact that the derivative of a constant function is zero.

Examples of antiderivatives: Look back at the list of derivatives of some important functions (lecture 2, p.3).

1. Let $f(x) = \mathrm{e}^x$ with $-\infty < x < +\infty$. Then $F(x) = \mathrm{e}^x$ is an antiderivative of $f(x)$. Indeed, $F'(x) = (\mathrm{e}^x)' = \mathrm{e}^x = f(x)$.

2. Let $f(x) = x^n$ with n a given positive integer, and $-\infty < x < +\infty$. Then $F(x) = \frac{1}{n+1}x^{n+1}$ is an antiderivative of $f(x)$ because $F'(x) = \left(\frac{1}{n+1}x^{n+1}\right)' = \frac{n+1}{n+1}x^{(n+1)-1} = x^n = f(x)$.

3. Show that an antiderivative of $\cos x$ is $\sin x$, and that an antiderivative of $\sin x$ is $-\cos x$.

4. Show that if $f(x) = x^b$ for a given $b \neq -1$, and for all $x > 0$, then $F(x) = \frac{1}{b+1}x^{b+1}$ is an antiderivative of f.

5. Show that if $f(x) = a^x$ for a given $a > 0$ but $a \neq 1$, and for all $-\infty < x < +\infty$, then $F(x) = \frac{1}{\ln a}a^x$ is an antiderivative of f.

We can now answer the question raised above, courtesy of the following important result:

Fundamental Theorem of Calculus I (FTC-I):
Let $f(x)$ be continuous on an open interval containing the compact interval $[a, b]$.
If $F(x)$ is an antiderivative of f, then:

$$\int_a^b f(x)dx = F(b) - F(a)$$

Notice that to compute the integral of f from a to b, it suffices that we know an antiderivative F of f. The integral is then simply the difference of the value of the antiderivative evaluated at the right and left endpoint of $[a, b]$.

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Examples:

1. What is the area enclosed between the horizontal axis and $f(x) = \sin x$ for $0 \leq x \leq \frac{\pi}{2}$? Answer: First, since $\sin(x)$ is positive for all x in $[0, \pi/2]$, we know that the area will be given by the integral of $\sin(x)$ from $x = 0$ to $x = \pi$. We also know that $F(x) = -\cos x$ is an antiderivative of f. By FTC-I, there follows:

$$\int_0^{\frac{\pi}{2}} \sin x \, dx = -\cos\left(\frac{\pi}{2}\right) - (-\cos(0)) = 0 - (-1) = 1.$$

   Notice how quickly we arrived at this answer, and reflect on the work we would have had to do to calculate this area using the procedure outlined earlier, using the definition of the integral as a limit!

2. Find

$$\int_0^1 (1-2x)\, \mathrm{e}^{x-x^2} \, dx$$

   Notice that $F(x) = \mathrm{e}^{x-x^2}$ is an antiderivative of $f(x) = (1-2x)\,\mathrm{e}^{x-x^2}$. Indeed, $F'(x) = \left(\mathrm{e}^{x-x^2}\right)' = (1-2x)\,\mathrm{e}^{x-x^2} = f(x)$ (we used the Chain Rule), and by the FTC-I:

$$\int_0^1 (1-2x)\, \mathrm{e}^{x-x^2} \, dx = \mathrm{e}^{1-1^2} - \mathrm{e}^{0-0^2} = 1 - 1 = 0.$$

The last example highlights a problem: Although it is very easy to calculate the integral of f from a to b, if an antiderivative of f is known, it may be very difficult to find an antiderivative, raising the question:

How do we find antiderivatives?

We will first develop 2 techniques, known as **Substitution, and Integration by Parts** to find antiderivatives, and we will see that each of these is related to one of the Derivative Rules, namely the Chain Rule, and the Product Rule respectively. If time permits, we will discuss other methods to find antiderivatives. For instance, we may discuss the method of partial fractions which is used to find the antiderivatives of rational functions.

Before proceeding, we introduce some new notation and new vocabulary. Suppose that $F(x)$ is an antiderivative of a given function $f(x)$, then we introduce the new notation:

$$\int f(x)dx = F(x) + c,$$

where c is an arbitrary constant, and we call the expression on the left the **indefinite integral** of f. The indefinite integral consists of a family of antiderivatives of f, where each member of the family corresponds to a specific value of the constant c. Notice that this new notation is very similar to $\int_a^b f(x)dx$, except that there are no integration bounds a or b. From now on, we also refer to $\int_a^b f(x)dx$ as a **definite**

integral of the function f (earlier on, we called it the integral of $f(x)$ from $x = a$ to $x = b$).

Substitution. Recall the Chain Rule for differentiable functions f and g:

$$(f(g(x)))' = f'(g(x))g'(x)$$

Another way of saying this is that $f(g(x))$ is an antiderivative of the function on the right. Equivalently, the definite integral of the function on the right is given by $f(g(x)) + c$, where c is an arbitrary constant:

$$\int f'(g(x))g'(x)dx = f(g(x)) + c. \tag{1}$$

The terminology *substitution* comes from the fact that the above equation can be rewritten as the "trivial" statement[3]:

$$\int f'(u)du = f(u) + c, \tag{2}$$

provided that we make the substitution:

$$u = g(x).$$

This is possible using the following formal argument: Taking derivatives with respect to x in the equation $u = g(x)$, yields:

$$\frac{du}{dx} = g'(x)$$

The **formal step** is that we interpret the derivative $\frac{du}{dx}$ on the left as a fraction, multiply both sides of the equation by the denominator dx of this fraction, which leads to:

$$du = g'(x)dx.$$

Together with the substitution equation $u = g(x)$, we see that (1) can be recast as (2), as claimed.

[3]We call this statement "trivial" because it expresses something self-evident. Namely, $f(u)$ is an antiderivative of $f'(u)$. Indeed, the derivative of $f(u)$ equals $f'(u)$.

lect12 10/18, (midterm 1 on 10/20), lect13 10/23

To see how the Substitution method works in practice, we work some:

Examples:

1. Find
$$\int \sin(x^2+1)xdx.$$
Letting
$$u = x^2+1,$$
yields that
$$du = 2xdx,$$
and hence
$$\int \sin(x^2+1)xdx = \int \sin(u)\frac{1}{2}du = -\frac{1}{2}\cos(u)+c = -\frac{1}{2}\cos(x^2+1)+c,$$
where in the 2nd equation we used that $-(1/2)\cos(u)$ is an antiderivative of $(1/2)\sin(u)$, and in the 3rd equation we switched back from the variable u to the variable x using the substitution $u = x^2+1$.

2. Find
$$\int \tan x dx$$
First, note that we can re-write this integral as
$$\int \frac{\sin x}{\cos x}dx$$
It is tempting to set
$$u = \sin x$$
Then
$$du = \cos x dx,$$
but this substitution does not lead to an obvious simplification of the integral.

So we try a perhaps less obvious substitution instead:
$$u = \cos x$$
Then
$$du = -\sin x dx,$$
and hence
$$\int \frac{\sin x}{\cos x}dx = \int \frac{-1}{u}du = -\ln|u|+c = -\ln|\cos x|+c$$
Here , we used the fact that $(\ln|u|)' = \frac{1}{u}$ for all $u \neq 0$. We know that this is true for all $u > 0$ from our list of derivatives of important functions in Part I of these notes (see p.3). To see that this is also true for $u < 0$, note that then $\ln|u| = \ln(-u)$, and therefore $(\ln|u|)' = \frac{1}{-u}.(-1) = \frac{1}{u}$ where we used the Chain Rule.

This example alerts us that choosing a "good" substitution may be tricky. Sometimes, an obvious choice for a substitution simply won't work, but a less obvious one, does.

3. Let's return to the sample question faced by Dr Fishburne who studies the migration patterns of great whites. Recall that the speed of a tagged shark over a single day, measured in mi/h was given by:

$$v(t) = \begin{cases} 2,\ 0 \le t \le 6 \\ 2 + \frac{1}{2}(t-6)^2,\ 6 < t \le 12 \\ 2 + \frac{1}{2}(t-18)^2,\ 12 < t \le 18 \\ 2,\ 18 < t \le 24 \end{cases}$$

where t is measured in hours. What is the total distance and the average speed of the shark during this day?

Let $x(t)$ denote the distance traveled by the shark by time t. Then clearly $x(0) = 0$ (it can't cover any distance in no time!), and our goal is to find the value of $x(24)$, the total distance covered by the shark in 1 day.

The fundamental relationship between $x(t)$ and $v(t)$ is:

$$x'(t) = v(t),$$

expressing that the instantaneous rate of change of the distance covered by the shark, is equal to its instantaneous speed. To find $x(24)$, we integrate both functions in this equation from $t = 0$ to $t = 24$:

$$\int_0^{24} x'(t)dt = \int_0^{24} v(t)dt$$

The integral on the left is equal to $x(24) - x(0)$ by virtue of the FTC-I (because of the trivial observation that $x(t)$ is an antiderivative of $x'(t)$), and therefore:

$$\int_0^{24} x'(t)dt = x(24) - x(0) = x(24) - 0 = x(24) = \int_0^{24} v(t)dt.$$

In other words, the total distance covered by the shark during this day is equal to the integral of its velocity from $t = 0$ to $t = 24$:

$$x(24) = \int_0^{24} v(t)dt.$$

We evaluate the last integral by using the function $v(t)$ defined earlier. Since $v(t)$ is given by 4 distinct functional forms, depending on the value of time t, we first split this integral in 4 parts, which is possible thanks to property 3 of integrals listed above:

$$\begin{aligned} \int_0^{24} v(t)dt &= \int_0^{6} v(t)dt + \int_6^{12} v(t)dt + \int_{12}^{18} v(t)dt + \int_{18}^{24} v(t)dt \\ &= \int_0^{6} 2dt + \int_6^{12} 2 + \frac{1}{2}(t-6)^2 dt + \int_{12}^{18} 2 + \frac{1}{2}(t-18)^2 dt + \int_{18}^{24} 2dt \\ &= 2(6-0) + \int_6^{12} 2 + \frac{1}{2}(t-6)^2 dt + \int_{12}^{18} 2 + \frac{1}{2}(t-18)^2 dt + 2(24-18), \end{aligned}$$

where the first and last integral are easily evaluated since the functions being integrated are constant functions. Using the 2nd property of integrals, we can find the 2nd integral:

$$\int_6^{12} 2 + \frac{1}{2}(t-6)^2 dt = \int_6^{12} 2dt + \int_6^{12} \frac{1}{2}(t-6)^2 dt = 2(12-6) + \int_6^{12} \frac{1}{2}(t-6)^2 dt$$

To find the last integral on the right, we will find the indefinite integral

$$\int \frac{1}{2}(t-6)^2 dt$$

using the substitution:

$$u(t) = t - 6 \;\Rightarrow\; u' = 1 \;\Rightarrow\; du = dt,$$

from which:

$$\int \frac{1}{2}(t-6)^2 dt = \int \frac{1}{2}u^2 du = \frac{1}{2}\int u^2 du = \frac{1}{2}\frac{u^3}{3} + c = \frac{u^3}{6} + c = \frac{(t-6)^3}{6} + c,$$

where we used the 1st property of integrals, and the fact that $u^3/3$ is an antiderivative of u^2. We can now use FTC-I to find:

$$\int_6^{12} \frac{1}{2}(t-6)^2 dt = \frac{(12-6)^3}{6} - \frac{(6-6)^3}{6} = 6^2 = 36,$$

and this let's us evaluate the 2nd integral:

$$\int_6^{12} 2 + \frac{1}{2}(t-6)^2 dt = 12 + 36 = 48.$$

In a similar way we can find the 3rd integral (Work out all the details!)[4]:

$$\int_{12}^{18} 2 + \frac{1}{2}(t-18)^2 dt = 48.$$

Putting it all together shows that

$$x(24) = \int_0^{24} v(t)dt = 12 + 48 + 48 + 12 = 120.$$

In other words, the shark covered a distance of 120mi in 1 day. It's average speeds is therefore 120mi/24h=5mi/h.

[4]Actually, it is instructive to plot the graph of $v(t)$, and to notice that this graph is symmetrical with respect to the vertical axis $t = 12$. Interpreting the 2nd integral and 3rd integral as the areas between the horizontal axis and the graph of $v(t)$ for t in $[6, 12]$ and $[12, 18]$ respectively, we see that these areas must be equal.

lect14 10/25 Here we introduce a second important technique to find antiderivatives of functions:

Integration by Parts. This technique is related to the Product Rule of derivatives.

Important remark: The text contains many crucial typos on this topic (Section 23.2), so please study this from these lecture notes only!

Recall the Product Rule for derivatives:

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x),$$

which is another way of saying that:

$$\int f'(x)g(x) + f(x)g'(x) = f(x)g(x) + c.$$

By the properties of integrals, we can rewrite this as:

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx + c \tag{3}$$

As in the Substitution Method discussed earlier, we now make the following substitutions:

$$u = f(x) \text{ and } v = g(x).$$

These imply formally that:

$$du = f'(x)dx \text{ and } dv = g'(x)dx,$$

which allows us to rewrite (3) more compactly as:

$$\int udv = uv - \int vdu + c. \tag{4}$$

This is known as the Integration-by-parts-formula, although usually one sets $c = 0$.

The key problem when applying the Integration by Parts method is to choose appropriate functions $u = f(x)$ and $v = g(x)$ (the "parts" from which the method derives its name). Let's see how this works in practice.

Example 1: Find

$$\int \ln|x|dx.$$

A natural choice for u and v is:

$$u = \ln|x| \text{ and } v = x,$$

because then

$$du = \frac{1}{x}dx \text{ and } dv = dx,$$

so that we can write

$$\begin{aligned}
\int \ln|x|dx &= \int udv = uv - \int vdu + c \\
&= x\ln|x| - \int x.\frac{1}{x}dx + c \\
&= x\ln|x| - \int 1dx + c \\
&= x\ln|x| - x + c.
\end{aligned}$$

Example 2: Find

$$\int x\,\mathrm{e}^{2x}\,dx$$

Let

$$u = x \text{ and } v = \mathrm{e}^{2x}\,.$$

Then

$$du = dx \text{ and } dv = 2\,\mathrm{e}^{2x}\,dx,$$

from which:

$$\int x\,\mathrm{e}^{2x}\,dx = \frac{1}{2}\int udv = \frac{1}{2}\left(uv - \int vdu\right) + c = \frac{1}{2}\left(x\,\mathrm{e}^{2x} - \int \mathrm{e}^{2x}\,dx\right) + c \qquad (5)$$

We now need to find:

$$\int \mathrm{e}^{2x}\,dx$$

This integral can be found using the Substitution Method by picking:

$$u = 2x, \text{ and } du = 2dx,$$

so that:

$$\int \mathrm{e}^{2x}\,dx = \frac{1}{2}\int \mathrm{e}^{u}\,du = \frac{1}{2}\,\mathrm{e}^{u} + c = \frac{1}{2}\,\mathrm{e}^{2x} + c \qquad (6)$$

Plugging (6) into (5):

$$\int x\,\mathrm{e}^{2x}\,dx = \frac{x\,\mathrm{e}^{2x}}{2} - \frac{\mathrm{e}^{2x}}{4} + c = \frac{\mathrm{e}^{2x}}{2}\left(x - \frac{1}{2}\right) + c$$

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Example 3: Sometimes one needs to perform integration by parts more than once. Here we discuss an example where it needs to be done twice. Find

$$\int \mathrm{e}^x \cos x dx$$

One reasonable choice is:

$$u = \mathrm{e}^x \text{ and } v = \sin x.$$

Then

$$du = \mathrm{e}^x \, dx \text{ and } dv = \cos x dx.$$

Then we can rewrite:

$$\int \mathrm{e}^x \cos x dx = \int u dv = uv - \int v du + c = \mathrm{e}^x \sin x - \int \sin x \, \mathrm{e}^x \, dx + c \qquad (7)$$

We notice that we end up with another indefinite integral on the right, namely:

$$\int \sin x \, \mathrm{e}^x \, dx.$$

We wish to rewrite this integral in the form of $\int \tilde{u} d\tilde{v}$. A natural choice to achieve this is by picking:

$$\tilde{u} = \mathrm{e}^x \text{ and } \tilde{v} = \cos x,$$

from which

$$d\tilde{u} = \mathrm{e}^x \, dx \text{ and } d\tilde{v} = -\sin x dx$$

Then

$$\int \sin x \, \mathrm{e}^x \, dx = -\int \tilde{u} d\tilde{v} = -\left(\tilde{u}\tilde{v} - \int \tilde{v} d\tilde{u}\right) = -\mathrm{e}^x \cos x + \int \cos x \, \mathrm{e}^x \, dx + c \quad (8)$$

We can now substitute (8) into (7) and obtain:

$$\begin{aligned} \int \mathrm{e}^x \cos x dx &= \mathrm{e}^x \sin x - \left(-\mathrm{e}^x \cos x + \int \mathrm{e}^x \cos x dx\right) + c \\ &= \mathrm{e}^x (\sin x + \cos x) - \int \mathrm{e}^x \cos x dx + c \end{aligned}$$

Notice that in the last step the integral we're trying to find, $\int \mathrm{e}^x \cos x dx$ appears with a minus sign. We can add this integral to both sides of the equation, yielding:

$$2 \int \mathrm{e}^x \cos x dx = \mathrm{e}^x (\sin x + \cos x) + c$$

Dividing by 2, gives us the result we're after:

$$\int \mathrm{e}^x \cos x dx = \frac{\mathrm{e}^x}{2} (\sin x + \cos x) + c$$

Remark: To confirm that our result makes sense, we could check that the function on the right, is indeed an antiderivative of the function $\mathrm{e}^x \cos x$. To do this, we calculate its derivative using the Product Rule:

$$\left(\frac{\mathrm{e}^x}{2}(\sin x + \cos x)\right)' = \frac{\mathrm{e}^x}{2}(\sin x + \cos x) + \frac{\mathrm{e}^x}{2}(\cos x - \sin x) = \mathrm{e}^x \cos x,$$

as expected.

Challenge problem: Show that for given nonzero numbers a and b:

$$\int \mathrm{e}^{ax} \cos(bx) dx = \frac{\mathrm{e}^{ax}}{a^2 + b^2}(b \sin(bx) + a \cos(bx)) + c$$

In fact, this formula remains valid if $a^2 + b^2 \neq 0$, that is, if at least one of these numbers is not zero.

Example 4: (adapted from Ex 23.5 in the text) Let $M(t)$ be the biomass (in g) of a certain tree frog at time $t \geq 0$ (where t is measured in years). Assume that the instantaneous growth rate of this tree frog is given by:

$$\frac{dM(t)}{dt} = (1 + t^2)\,\mathrm{e}^{-t}\,.$$

If $M(0) = 3$ g, what is $M(1)$, the tree frog's biomass after 1 year? By what percentage did it increase its initial biomass?

To answer this question, note first that since $M(t)$ is an antiderivative of $dM(t)/dt$, the FTC implies that:

$$\int_0^1 \frac{dM(t)}{dt} dt = M(1) - M(0),$$

and therefore, by solving for $M(1)$, using that $M(0) = 3$ (dropping the unit of g), and the above formula for $dM(t)/dt$, that:

$$M(1) = 3 + \int_0^1 (1 + t^2)\,\mathrm{e}^{-t}\,dt \tag{9}$$

To calculate the definite integral on the right, we first attempt to find an antiderivative for the function $(1 + t^2)\,\mathrm{e}^{-t}$:

$$\int (1 + t^2)\,\mathrm{e}^{-t}\,dt,$$

using the Integration by Parts Method. Let

$$u = 1 + t^2 \text{ and } v = \mathrm{e}^{-t}\,.$$

Then

$$du = 2tdt \text{ and } dv = -\,\mathrm{e}^{-t}\,dt$$

and thus:

$$\int (1 + t^2)\,\mathrm{e}^{-t}\,dt = -\int udv = -\left(uv - \int vdu\right) = -(1 + t^2)\,\mathrm{e}^{-t} + 2\int \mathrm{e}^{-t}\,tdt \tag{10}$$

Let's now consider

$$\int \mathrm{e}^{-t}\, t dt.$$

Choosing

$$\tilde{u} = t \text{ and } \tilde{v} = \mathrm{e}^{-t},$$

we also have that

$$d\tilde{u} = dt \text{ and } d\tilde{v} = -\,\mathrm{e}^{-t}\, dt,$$

so that:

$$\begin{aligned}
\int \mathrm{e}^{-t}\, t dt &= -\int \tilde{u} d\tilde{v} = -\left(\tilde{u}\tilde{v} - \int \tilde{v} d\tilde{u}\right) + c \\
&= -t\,\mathrm{e}^{-t} + \int \mathrm{e}^{-t}\, dt + c \\
&= -t\,\mathrm{e}^{-t} - \mathrm{e}^{-t} + c \\
&= -(t+1)\,\mathrm{e}^{-t} + c
\end{aligned}$$

Plugging this back into (10), yields:

$$\int (1+t^2)\,\mathrm{e}^{-t}\, dt = -(1+t^2)\,\mathrm{e}^{-t} - 2(t+1)\,\mathrm{e}^{-t} + c = -(t^2+2t+3)\,\mathrm{e}^{-t} + c$$

In other words, the function $-(t^2+2t+3)\,\mathrm{e}^{-t}$ is an antiderivative of the function $(1+t^2)\,\mathrm{e}^{-t}$. Returning to (9), and applying the FTC, we find that:

$$\begin{aligned}
M(1) &= 3 + \int_0^1 (1+t^2)\,\mathrm{e}^{-t}\, dt \\
&= 3 + \left(-(1^2+2.1+3)\,\mathrm{e}^{-1} - (-(0^2+2.0+3)\,\mathrm{e}^{-0})\right) \\
&= 3 + \left(-6\,\mathrm{e}^{-1} + 3\right) = 6(1-\mathrm{e}^{-1})
\end{aligned}$$

To conclude, the biomass $M(1)$ of the tree frog after 1 year is $6(1-\mathrm{e}^{-1})$ g ≈ 3.80 g. In other words, the tree frog has grown by $(3.80-3)/3$ %, or about 27% of its initial biomass during this year.

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Improper integrals: Before we move on to yet another application of integrals (namely, continuous random variables), we introduce the notion of an improper integral. There are different kinds of improper integrals. Either the function being integrated is undefined in some point of the integration interval. Or, the integration interval is not a bounded interval. We shall only discuss improper integrals of the latter kind, and start with an example. Let $z > 1$ be an arbitrary real number, and consider

$$\int_1^z \frac{1}{x^2} dx.$$

We know that $x^{-1}/(-1)$ is an antiderivative of $1/x^2$, and thus the FTC tells us that:

$$\int_1^z \frac{1}{x^2} dx = \frac{z^{-1}}{-1} - \frac{1^{-1}}{-1} = 1 - \frac{1}{z}.$$

Now notice that if we let $z \to \infty$, we get that:

$$\lim_{z\to\infty} \int_1^z \frac{1}{x^2} dx = \lim_{z\to\infty} 1 - \frac{1}{z} = 1.$$

This limit exists (it equals 1), and instead of writing the limit on the left-hand side, we replace it by the shorthand notation:

$$\int_1^\infty \frac{1}{x^2} dx,$$

which is an example of an **improper integral**.

More generally, when given a continuous function $f(x)$, defined on an interval $[a, \infty)$, for a given real number a, we call

$$\lim_{z\to\infty} \int_a^z f(x)dx$$

an improper integral, and it exists provided this limit exists. In this case, we denote it by:

$$\int_a^\infty f(x)dx \tag{11}$$

In some cases, the lower bound grows negative and unbounded and then we have an improper integral which is denoted by:

$$\int_{-\infty}^a f(x)dx = \lim_{z\to-\infty} \int_z^a f(x)dx. \tag{12}$$

Finally, in some cases both lower and upper bound grow unbounded, and then we write:

$$\int_{-\infty}^\infty f(x)dx \tag{13}$$

In the above notation it is typically assumed that the lower and upper bound approach their limits at the same rate:

$$\lim_{z\to\infty} \int_{-z}^z f(x)dx$$

We remark that when $f(x)$ is a non-negative function, then the geometrical interpretation of the improper integrals (11), (12) and (13) are that they represent the areas above the horizontal axis and below the graph of $f(x)$ in the respective intervals $[a, \infty)$, $(-\infty, a]$ and $(-\infty, \infty)$. For many functions these areas are not finite (equivalently, these areas are not finite). For instance, if $f(x) = 1$ for all x in $(-\infty, \infty)$, then none of these 3 improper integrals exist. Indeed, none of the 3 corresponding limits exist, e.g.

$$\lim_{z\to\infty} \int_a^z 1dx = \lim_{z\to\infty} z - a = \infty,$$

which is not finite.

From our first example in the beginning of this lecture, it would be tempting to think that if a function $f(x)$ decreases to zero when x approaches infinity, then the improper integral would exist. But this is not necessarily true. For instance, if $f(x) = 1/x$, and $z > 1$, then by the FTC:

$$\int_1^z \frac{1}{x} dx = \ln(|z|) - \ln(1) = \ln z,$$

because $F(x) = \ln|x|$ is an antiderivative of $f(x) = 1/x$. But

$$\lim_{z\to\infty} \int_1^z \frac{1}{x} dx = \lim_{z\to\infty} \ln z = \infty,$$

and since this limit is not finite, the improper integral does not exist. The reason why the area under the graph of $f(x) = 1/x$ is not finite, is because the function $1/x$ approaches zero too slowly as x approaches ∞. Contrast this to our initial example of the function $f(x) = 1/x^2$, whose improper integral over the interval $[1, \infty)$ was equal to 1. The difference between the two functions $1/x$ and $1/x^2$ is that the former does not approach zero fast enough as x approaches ∞, whereas the latter function does.

Example: Let $\lambda > 0$ be a given parameter. Show that the improper integral

$$\int_0^\infty \lambda\, \mathrm{e}^{-\lambda t}\, dt$$

exists.

To show this, we start by letting $z > 0$ be a real number, and we examine:

$$\int_0^z \lambda\, \mathrm{e}^{-\lambda t}\, dt. \tag{14}$$

Our first goal is to find an antiderivative of the function $\lambda\, \mathrm{e}^{-\lambda t}$:

$$\int \lambda\, \mathrm{e}^{-\lambda t}\, dt.$$

Pick the natural Substitution:

$$u = \lambda t \;\Rightarrow du = \lambda dt,$$

and we find that

$$\int \lambda\, \mathrm{e}^{-\lambda t}\, dt = \int \mathrm{e}^{-u}\, du = -\,\mathrm{e}^{-u} + c = -\,\mathrm{e}^{-\lambda t} + c \tag{15}$$

where we used the fact that $F(u) = -\,\mathrm{e}^{-u}$ is an antiderivative of the function $f(u) = \mathrm{e}^{-u}$ (indeed, $F'(u) = f(u)$ by an easy calculation of the derivative using the Chain Rule). We can now use (15), and the FTC to compute (14):

$$\int_0^z \lambda\, \mathrm{e}^{-\lambda t}\, dt = -\,\mathrm{e}^{-\lambda z} - \left(-\,\mathrm{e}^{-\lambda 0}\right) = 1 - \mathrm{e}^{-\lambda z}$$

Now we let z approach infinity, and find that

$$\int_0^\infty \lambda\, \mathrm{e}^{-\lambda t}\, dt = \lim_{z\to\infty} \int_0^z \lambda\, \mathrm{e}^{-\lambda t}\, dt = \lim_{z\to\infty} 1 - \mathrm{e}^{-\lambda z} = 1 - 0 = 1,$$

where we used the key fact that $\lambda > 0$ to calculate the last limit. We conclude that the improper integral

$$\int_0^\infty \lambda\, \mathrm{e}^{-\lambda t}\, dt = 1 \tag{16}$$

exists. We shall encounter this improper integral again soon, after we introduce the new concept of *continuous random variables* and discuss a very important example, known as an *exponentially distributed random variable.*

Exercise 1: We have already seen that $\int_1^\infty 1/x dx$ does not exist, but that $\int_1^\infty 1/x^2 dx$ does. What about the existence of

$$\int_1^\infty \frac{1}{x^p} dx,$$

when p is a parameter assumed to satisfy $p > 1$? Show that in this case the improper integral always exists, and calculate its value.

Exercise 2: Determine if the following improper integral exists, and if so, calculate its value:

$$\int_0^\infty x\, \mathrm{e}^{-x^2}\, dx$$

Part III: Continuous Random Variables

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Many quantities encountered in the life sciences, but also in other sciences, exhibit randomness. For instance:

1. The length of salamanders ranges from 2.7cm to a whopping 1.8m for the Chinese giant salamander (the size of a tall man!). See Wikipedia.

2. The weight of sexually mature buffalos ranges between 318kg to 1000kg. See Wikipedia.

3. The annual rainfall in Oregon ranges between 5in in the Alvord Desert in southeastern Oregon to 200in in some coastal regions. See Wikipedia.

4. The waiting time experienced by the barista at Starbucks before the next customer shows up could be any non-negative time.

In all these cases, the quantities of interest (length in cm, weight in kg, rainfall in inches, and time in min) take on real values, and they serve as examples of **continuous random variables**.

This is in contrast to **discrete random variables**, which only take on values in a *discrete set*, meaning that their value belongs to either a finite set, or to a countable set. An example of a discrete random variable is a coin toss which can either be heads or tails, so there are only 2 possible values $\{H, T\}$. Another example of a discrete random variable, but one having a countable set of outcomes, is the number of coin tosses we should perform to get heads for the first time. This number could be any number in the set $\{1, 2, 3, \dots\}$, i.e. the set of positive integers.

In these notes the discussion will be restricted to continuous random variables, but the mathematical development for discrete random variables is entirely analogous.

To begin a more formal treatment, we denote a continuous random variable by X. The set of all possible values X can take on is denoted as S. In the 4 examples above, we have that S are intervals: $S = [2.7, 180]$ for the length of salamanders in cm, $S = [318, 1000]$ for the weight of buffalo in kg, $S = [5, 200]$ for the rainfall in inches in Oregon, and $S = [0, \infty)$ for the waiting time experienced by the Starbucks barista.

Now it is very natural to ask the following question:

How likely is it that X takes on values in a given subset S' of S?

For example, we could ask how likely it is that the length of a salamander is between 10cm and 20cm. In this case, we would have that the subset S' is the interval $[10, 20]$. Or, we could ask how likely it is that the rainfall in a Oregon is less than 50in. In this case, the subset S' is $[5, 50)$. Or, we could ask how likely it is that the barista has to wait at least 10min for the next costumer to show up. In this case $S' = [10, \infty)$.

Motivated by these examples -and also to avoid technical complications later- we shall assume that S is a (possibly infinite) interval, and restrict the subsets S' to be **intervals, or finite unions of non-intersecting intervals**.

To answer the question above, we associate to the continuous random variable X its **probability density function (or pdf for short)** $f(x)$, where x belongs to the set of possible outcomes S. The pdf $f(x)$ should be continuous and non-negative, and such that for all subsets S' of S:

$$P(X \text{ belongs to } S') = \int_{S'} f(x)dx \tag{17}$$

This formula needs further explanation. First, the expression on the left stands for "the probability that the continuous random variable X belongs to the subset S'". The equation merely states that his probability is given by the integral of the probability density function $f(x)$, over the set S'. Recall that we restricted S' to be an interval, or a finite union of intervals. We need to clarify what the integral on the right means. Suppose that S' is an interval, say $S' = [a, b]$ for given $a < b$. Then the integral on the right is just the definite integral of $f(x)$ over $[a, b]$:

$$\int_{S'} f(x)dx = \int_a^b f(x)dx.$$

If S' is a finite union of n non-intersecting intervals, say of the intervals $[a_1, b_1]$, $[a_2, b_2]$, $\ldots$, $[a_n, b_n]$, then the integral on the right is the sum of the n definite integrals of $f(x)$ over each of these intervals:

$$\int_{S'} f(x)dx = \int_{a_1}^{b_1} f(x)dx + \int_{a_2}^{b_2} f(x)dx + \cdots + \int_{a_n}^{b_n} f(x)dx.$$

Formula (17) shows that we can answer the question raised earlier, once we are given the pdf $f(x)$ of the random variable X. We can also think geometrically about this formula: since $f(x)$ is non-negative, its graph is above the horizontal axis, and therefore the probability that the random variable belongs to $[a, b]$ say, is simply the area below the graph of $f(x)$, and above the horizontal axis, when x ranges between a and b. And it is the sum of such areas when the set S' consists of a union of intervals.

Before discussing several important examples of continuous random variables, we point out an important property of pdf's, which follows immediately when letting $S' = S$ in (17):

$$\int_S f(x)dx = 1. \tag{18}$$

Indeed, the probability that X takes on values in the set S, must equal 1 (recall that S is defined as the set of *all* possible values that X can take on). This is an almost trivial statement, but it imposes an important constraint on the functions $f(x)$ that can serve as pdf's.

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Important examples of continuous random variables:

1. **Uniform random variable**. Let $a < b$ be given. We say that X is a uniform random variable if
$$S = [a, b] \text{ and } f(x) = \frac{1}{b-a}$$

2. **Exponential random variable**. Let $\lambda > 0$ be given. We say that X is an exponential random variable (with parameter λ) if
$$S = [0, \infty) \text{ and } f(x) = \lambda\, \mathrm{e}^{-\lambda x}$$

3. **Normal random variable**. Let μ and $\sigma > 0$ be given (these are called the *mean* and *standard deviation* respectively). We say that X is a normal random variable (with parameters μ and σ) if
$$S = (-\infty, \infty) \text{ and } f(x) = \frac{1}{\sqrt{2\pi}\sigma}\, \mathrm{e}^{-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2}$$

Let us first verify that these pdf's satisfy the constraint in (18):

1. $$\int_a^b \frac{1}{b-a} dx = \frac{1}{b-a}\int_a^b 1 dx = \frac{1}{b-a}(b-a) = 1,$$
where we used the FTC, and the fact that the function x is an antiderivative of the function 1: $(x)' = 1$.

2. $$\int_0^\infty \lambda\, \mathrm{e}^{-\lambda x}\, dx.$$
First, recall that this is an improper integral. In fact, we calculated its value in the previous lecture and showed that it equals 1, see (16) (but note that there we used t as the integration variable, rather than x).

3. $$\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}\sigma}\, \mathrm{e}^{-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2}\, dx$$
Actually, calculating this improper integral requires techniques that go beyond what we've learned in this course so far. In particular, it would require integration of functions that depend on 2 variables, and so we will not attempt to calculate this integral here. But, it is possible to show that it equals 1, as it should.

Example 1: Let X be a continuous random variable with $S = [0, 1]$, and pdf
$$f(x) = k(x-1)^2, \text{ for } x \text{ in } S.$$
Find the value of the constant k that ensures that $f(x)$ is a pdf. Then find
$$P(X \le 0.5) \text{ and } P(X > 0.5)$$

Solution: To ensure that $f(x)$ is a pdf, we must have that

$$\int_S f(x)dx = \int_0^1 k(x-1)^2 dx = 1.$$

Now using the substitution

$$u = x - 1, \; du = dx$$

we can find the antiderivative of

$$\int k(x-1)^2 dx = k \int u^2 du = k\frac{u^3}{3} + c = \frac{k}{3}(x-1)^3 + c \tag{19}$$

Then by the FTC:

$$\int_0^1 k(x-1)^2 dx = \frac{k}{3}(1-1)^3 - \left(\frac{k}{3}(0-1)^3\right) = \frac{k}{3},$$

and since we want that the latter equals 1, we must have that

$$k = 3.$$

Next we calculate

$$\begin{aligned} P(X \leq 0.5) &= P(X \text{ belongs to } [0, 0.5]) \\ &= \int_0^{0.5} f(x)dx \\ &= \int_0^{0.5} 3(x-1)^2 dx \\ &= (0.5-1)^3 - (0-1)^3 = -0.5^3 + 1 = -\frac{1}{8} + 1 = \frac{7}{8} \end{aligned}$$

where we have used (19) with $k = 3$ and applied the FTC to calculate the last definite integral.

To find $P(X > 0.5)$, we could proceed as above in the calculation of $P(X \leq 0.5)$. But, there is a short-cut here! Recall from the previous course MTH 227, that if A and B are disjoint events (meaning that the intersection of the sets A and B is empty), then

$$P(A \cup B) = P(A) + P(B) \tag{20}$$

We claim that we can apply this formula here to calculate $P(X > 0.5)$. First, let's define the two events A and B:

$$A = \{X \leq 0.5\}, \text{ and } \{B = X > 0.5\}.$$

We note that then A and B are disjoint, and applying (20), we get that:

$$P(\{X \leq 0.5\} \text{ or } \{X > 0.5\}) = P(X \leq 0.5) + P(X > 0.5)$$

Now notice that the probability on the left side is simply $P(X \text{ belongs to } [0, 1])$, which is 1. And we have already calculated $P(X \leq 0.5) = 7/8$. Consequently, we can solve for $P(X > 0.5)$:

$$P(X > 0.5) = 1 - \frac{7}{8} = \frac{1}{8}.$$

We can even avoid the above refresher about MTH 227, if we think geometrically: $P(X \leq 0.5)$ is the area under the graph of $f(x)$ and the horizontal axis for x between 0 and 0.5. Similarly, $P(X > 0.5)$ is the area under the graph of $f(x)$ and the horizontal axis for x between 0.5 and 1. But the total area when x ranges from 0 to 1 is equal to 1, which justifies the last step immediately.

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Example 2: Suppose that X is a random variable with range $S = [-1, 1]$ and pdf

$$f(x) = k(1 - |x|), \text{ for } x \text{ in } S.$$

Find the value of k that ensures that $f(x)$ is a pdf. Then find $P(|X| \leq 0.5)$.

Solution: To find k, we impose that:

$$\int_S f(x)dx = \int_{-1}^{1} f(x)dx = 1$$

Since $|x| = x$ when $x > 0$, but $|x| = -x$ when $x < 0$, we break up the integral in two parts, depending on the sign of x:

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{0} f(x)dx + \int_{0}^{1} f(x)dx = k\int_{-1}^{0} 1-(-x)dx + k\int_{0}^{1} 1-xdx = 1. \quad (21)$$

Thus, we notice that we need the following two indefinite integrals:

$$\int 1 + xdx \text{ and } \int 1 - xdx.$$

These are pretty straightforward to calculate:

$$\int 1 + xdx = x + \frac{x^2}{2} + c \text{ and } \int 1 - xdx = x - \frac{x^2}{2} + c.$$

We can now use the FTC to calculate the two definite integrals in (21):

$$k\left(0 + \frac{0^2}{2} - (-1 + \frac{(-1)^2}{2})\right) + k\left(1 - \frac{1^2}{2} - (0 - \frac{0^2}{2})\right) = 1,$$

which simplifies to:

$$k\left(\frac{1}{2}\right) + k\left(\frac{1}{2}\right) = 1, \text{ or } k = 1.$$

Of course, we could have been a bit smarter, and avoided this entire calculation if we would have plotted the graph of $f(x)$. We would have noticed that the graph is tent-shaped, and the area under the graph when x varies from -1 to $+1$, is simply the area of a triangle with base $1 - (-1) = 2$, and height k. This equals $\frac{2k}{2}$, and since this are must equal 1, we see that k must equal 1 as well.

Next, we determine

$$P(|X| \leq 0.5).$$

First note that the statement that

$$|X| \leq 0.5,$$

is equivalent to the statement that

$$-0.5 \leq X \leq 5,$$

or to the statement that

$$X \text{ belongs to the subinterval } [-0.5, 0.5].$$

Therefore, we have that

$$P(|X| \leq 0.5) = P(X \text{ belongs to the subinterval } [-0.5, 0.5]) = \int_{-0.5}^{0.5} 1 - |x| dx,$$

where we used the formula (17) with $f(x) = 1 - |x|$. Let's be smart about determining the value of the last integral, and think geometrically: It's value equals the sum of the area of a rectangle and the area of a triangle. The rectangle has base $0.5 - (-0.5) = 1$ and height $f(0.5) = 0.5$, so its area is 0.5. The triangle has the same base 1, and height $1 - 0.5 = 0.5$, so its area is $1/4$. The sum of these two areas is $1/2 + 1/4$, and thus we conclude that:

$$P(|X| \leq 0.5) = \frac{3}{4} = 75\%.$$

Example 3 A Starbucks barista wants to know what kind of day it will be at work, waiting for customers at the counter. (S)he doesn't like it when there is little work, but also isn't too happy when too many customers show up. So (s)he asks: "What is the probability that I either have to wait less than 1 minute, or more than 10 minutes for the next customer to show up." The only information available is that the waiting time is an exponential random variable, with parameter $\lambda = 1$ per minute (this parameter always has to be a *rate*, meaning it must have units 1/time).

Solution: The key to solving this problem is to translate the word problem into a mathematical problem. Let X denote the waiting time for the next costumer. This is an exponential random variable with $S = [0, \infty)$ and with parameter $\lambda = 1$ (per minute). A waiting time of less than 1 minute corresponds to $X < 1$, and similarly, a waiting time of more than 10 minutes corresponds to $X > 10$. Thus, the question the barista is asking is what the value is of:

$$P(X < 1 \text{ or } X > 10).$$

We note that $[0, 1)$ and $(10, \infty)$ are non-intersecting subintervals of $S = [0, \infty)$, and therefore formula (17) can be applied to express the above question as:

$$P(X < 1 \text{ or } X > 10) = \int_0^1 f(x)dx + \int_{10}^{\infty} f(x)dx = \int_0^1 \mathrm{e}^{-x}\, dx + \int_{10}^{\infty} \mathrm{e}^{-x}\, dx,$$

where we used the fact that $f(x) = \lambda\, \mathrm{e}^{-\lambda x}$ with $\lambda = 1$. The first integral is a definite integral, and the second is an improper integral. In either case, we need an antiderivative of the function e^{-x}, but this is simply $-\mathrm{e}^{-x}$. Applying the FTC we can calculate the first definite integral:

$$\int_0^1 \mathrm{e}^{-x}\, dx = -\mathrm{e}^{-1} - \left(-\mathrm{e}^{-0}\right) = 1 - \mathrm{e}^{-1}.$$

Similarly, the second improper integral can be calculated:

$$\int_{10}^{\infty} \mathrm{e}^{-x}\, dx = \lim_{z \to \infty} \int_{10}^{z} \mathrm{e}^{-x}\, dx = \lim_{z \to \infty} -\mathrm{e}^{-z} - \left(-\mathrm{e}^{-10}\right) = -0 + \mathrm{e}^{-10} = \mathrm{e}^{-10}.$$

Thus:

$$P(X < 1 \text{ or } X > 10) = 1 - \mathrm{e}^{-1} + \mathrm{e}^{-10} \approx 63\%.$$

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Expected value of a continuous random variable Let X be a continuous random variable with range S and probability density function $f(x)$. The **expected value (or mean, or expectation) of** X, is defined as

$$E(X) = \int_S x f(x) dx \tag{22}$$

Examples:

1. Let X be a uniform random variable with $S = [a, b]$ for some given $a < b$, and $f(x) = 1/(b-a)$. The expected value of X is:

$$E(X) = \int_a^b x \frac{1}{b-a} dx = \frac{1}{b-a}\left(\frac{b^2}{2} - \frac{a^2}{2}\right) = \frac{(b-a)(b+a)}{2(b-a)} = \frac{a+b}{2},$$

   i.e. the expectation of a uniform random variable is **the average of the given numbers** a **and** b.

2. Let X be an exponential random variable with parameter $\lambda > 0$, range $S = [0, \infty)$ and $f(x) = \lambda\, \mathrm{e}^{-\lambda x}$. The expected value of X is:

$$E(X) = \int_0^\infty x\lambda\, \mathrm{e}^{-\lambda x}\, dx$$

   Let's calculate

$$\int x\lambda\, \mathrm{e}^{-\lambda x}\, dx$$

   using integration by parts:

$$u = x \text{ and } v = -\,\mathrm{e}^{-\lambda x},$$

   so that

$$du = dx \text{ and } dv = \lambda\, \mathrm{e}^{-\lambda x}\, dx$$

   Then

$$\begin{aligned}\int x\lambda\, \mathrm{e}^{-\lambda x}\, dx &= \int u dv = uv - \int v du = -x\, \mathrm{e}^{-\lambda x} + \int \mathrm{e}^{-\lambda x}\, dx = -x\, \mathrm{e}^{-\lambda x} - \frac{1}{\lambda}\, \mathrm{e}^{-\lambda x} + c \\ &= -\left(x + \frac{1}{\lambda}\right) \mathrm{e}^{-\lambda x} + c\end{aligned}$$

   There follows that

$$E(X) = \int_0^\infty x\lambda\, \mathrm{e}^{-\lambda x}\, dx = \lim_{z\to\infty} \int_0^z x\lambda\, \mathrm{e}^{-\lambda x}\, dx = \lim_{z\to\infty} -\left(z + \frac{1}{\lambda}\right) \mathrm{e}^{-\lambda z} + \left(0 + \frac{1}{\lambda}\right) \mathrm{e}^{-0} = \frac{1}{\lambda}.$$

   In other words, the expected value of an exponential random variable is **the reciprocal of its parameter** λ. For example, if the waiting time of the Starbucks barista is an exponential random variable with $\lambda = 1$ per minute, then the expected waiting time is 1 minute. If $\lambda = 2$ per minute, then the expected waiting time is 1/2 minute, or 30 seconds. Thus, the bigger λ is, the smaller the expectation of the waiting time will be. The parameter λ is sometimes called the **intensity** of the exponential random variable X, and our calculation has shown that if the intensity gets bigger, the expectation of the waiting time gets smaller.

3. Let X is a normal random variable with parameters μ and σ. Then

$$S = (-\infty, \infty) \text{ and } f(x) = \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2}$$

The expected value of X is:

$$E(X) = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2} dx$$

This looks like a very complicated improper integral to evaluate, but it turns out that it is not, using some clever manipulations. We first consider a different improper integral:

$$\int_{-\infty}^{\infty} (x-\mu) f(x) dx = \int_{-\infty}^{+\infty} (x-\mu) \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2} dx$$

Using the substitution $z = x - \mu$, hence $dz = dx$, and also $z \to \pm\infty$ when $x \to \pm\infty$, we can rewrite the last improper integral as:

$$\int_{-\infty}^{\infty} (x-\mu) f(x) dx = \int_{-\infty}^{+\infty} (x-\mu) \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-\left(\frac{x-\mu}{\sqrt{2}\sigma}\right)^2} dx = \int_{-\infty}^{+\infty} z\, \mathrm{e}^{-\left(\frac{z}{\sqrt{2}\sigma}\right)^2} dz$$

The key observation is that the integrand in the last improper integral is an **odd function** (i.e., for any z, the function evaluated in $-z$ is the opposite of the function evaluated at z). But then it follows that the improper integral must be zero! (Think geometrically, and notice that the improper integral can be written as the sum of two improper integrals, one from $-\infty$ to 0, and the other from 0 to $+\infty$. Because the integrand is an odd function, these two improper integrals represent two equal areas between the horizontal axis and the graph of the integrand; but one is counted positively and the other negatively, hence they add up to zero.) Thus, we have shown that:

$$\int_{-\infty}^{\infty} (x-\mu) f(x) dx = 0.$$

However, we can also split the integral in the left as:

$$0 = \int_{-\infty}^{\infty} (x-\mu) f(x) dx = \int_{-\infty}^{\infty} x f(x) dx - \mu \int_{-\infty}^{\infty} f(x) dx = E(X) - \mu.(1),$$

where we used the definition of the expectation of X, and the fact that $f(x)$ is a pdf, hence its integral over $S = (-\infty, +\infty)$ equals 1. This implies that

$$E(X) = \mu,$$

i.e. the expectation of a normal random variable is equal to the parameter μ.

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lect22 11/15

The memoryless property If X is an exponential random variable, then for any positive x and y:

$$P(X > x + y | P > y) = P(X > x) \tag{23}$$

To see this, recall the definition of conditional probability:

$$P(A|B) = \frac{P(A \cap B)}{P(B)},$$

where A and B are events in the space of outcomes, and $P(B) > 0$. Thus,

$$\begin{aligned}
P(X > x + y | P > y) &= \frac{P(X > x + y \text{ and } X > y)}{P(X > y)} \\
&= \frac{P(X > x + y)}{P(X > y)} \\
&= \frac{\int_{x+y}^{\infty} \lambda \, \mathrm{e}^{-\lambda t} \, dt}{\int_{y}^{\infty} \lambda \, \mathrm{e}^{-\lambda t} \, dt} \\
&= \frac{\mathrm{e}^{-\lambda(x+y)}}{\mathrm{e}^{-\lambda y}} \\
&= \mathrm{e}^{-\lambda x} \\
&= P(X > x)
\end{aligned}$$

[Make sure to verify the 4th equation, where the two improper integrals were evaluated!]

Property (23) is known as the **memoryless property** of exponential random variables. In the context of our example of the Starbucks barista who is waiting for the next customer, it expresses that **the distribution of this waiting time is independent of how long the barista has already waited**.

More precisely, $P(X > x)$ is the probability that the waiting time is larger than x, that is, the probability that the first customer has not yet shown up by time x. And $P(X > x + y | X > y)$ is the conditional probability, conditioned on the event that the customer has not shown up by time y, that the customer won't show up before time $x+y$, that is, won't show up until at least another x units of time elapse. The memoryless property says that these two probabilities are equal.

Actually, it can be shown that the only continuous random variables that have the memoryless property, are the exponential random variables.

As an example, let X be an exponential random variable with $\lambda = 1$ per minute. Find

$$P(X > 5 | X > 4).$$

In other words, find the probability that the waiting time is at least 5 minutes, conditioned on the event that it is at least 4 minutes. In practical terms, this is asking for the probability that the barista has to wait at least 5 minutes, knowing that (s)he has already waited for 4 minutes. By the memoryless property, this is:

$$P(X > 5 | X > 4) = P(X > 5 - 4) = P(X > 1) = \int_1^{\infty} \mathrm{e}^{-x} \, dx = \lim_{z \to \infty} -\mathrm{e}^{-z} - (-\mathrm{e}^{-1}) = \mathrm{e}^{-1} \approx 37\%$$

Note that $P(X > 5|X > 4)$ is **not** the same as

$$P(X > 5) = \int_5^\infty \mathrm{e}^{-x}\, dx = \lim_{z\to\infty} -\mathrm{e}^{-z} - (-\mathrm{e}^{-5}) = \mathrm{e}^{-5} \approx 0.7\%$$

1 Part IV: Linear algebra and modeling

lect 23, 11/20

Motivating example: Let us consider a population of **juveniles** and **adults** of some population which could be anything (fish, elephants, grizzly bears etc). We wish to track the number of juveniles and adults at specific times right after the reproductive season. We scale time, so that $t = 0, 1, 2, \ldots$ represent the times right after each reproductive season. We denote $J(t)$ and $A(t)$ as the number of juveniles and adults respectively at time t, the time right after the t-th reproductive season.

We must now **model** the progression through time of the variables $A(t)$ and $J(t)$. This requires a solid understanding of the biology of the population you are working with. We will keep the discussion simple here, but in real applications some of the ideas presented below may not hold, and may have to be modified. Here are the simple rules we propose to work with:

1. Only adults can reproduce, that is, juveniles cannot reproduce.
2. Each adult produces 0.5 offspring on average, and those offspring are added to the class of juveniles after the reproductive season. [5]
3. The probability that a juvenile survives to the next reproductive season is 50%, and when it does it is considered an adult, capable of reproduction later.
4. The probability that an adult survives to the next reproductive season is 100%. In other words, adults will live "forever".

These rules will enable us to find the composition of the population at time $t+1$, given its composition at time t, where t is any non-negative integer. In other words, we will predict or "project" the composition of the population at time $t+1$, given its composition at time t. Here is the mathematical model that encodes this prediction, based on the 4 rules given earlier. The model is valid for any $t = 0, 1, 2, \ldots$:

$$\begin{aligned} J(t+1) &= 0.5A(t) \\ A(t+1) &= 0.5J(t) + A(t) \end{aligned}$$

This model allows us to predict the value of the pair $J(t+1), A(t+1)$ if we know the value of the pair $J(t), A(t)$. A moment of reflection then shows that if we know the **initial population composition**, that is, if we are given a pair $J(0), A(0)$, then we should in principle be able to calculate the population composition $J(t), A(t)$ at any future time t, by recursively using the model. Indeed, given $J(0), A(0)$ we can use the model to compute $J(1), A(1)$. But then we can use the latter and the model once again, to compute $J(2), A(2)$. This process can be continued indefinitely. But how do we do this in an orderly fashion? And can we make any predictions about

[5] This may seem like a weird assumption, especially when you are considering a population that exhibits sexual reproduction, which requires the presence of males and females. However, a model that includes separate groups of male and female juveniles and adults would become quite complicated. We prefer to start with a simpler model here. On the other hand, many cellular populations do not exhibit sexual reproduction, but instead undergo cell division. Our model quite reasonably captures the dynamics of those types of cell populations. Alternatively, we could interpret $A(t)$ as the number of **female adults** in a sexually reproducing population.

what happens as $t \to \infty$? This is where **linear algebra** will provide us with some powerful tools and ideas.

Vectors, matrices, and operations on/between them

Matrices are simply rectangular arrays of numbers, delimited by round brackets. For instance:

$$\begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 6 \\ -1 & 0 & 0 \end{pmatrix}$$

In general a matrix has n rows and m columns. We say that the matrix is an "n by m" matrix, or an "$n \times m$" matrix. (The order of n and m is important in these expressions!) For instance, the first matrix above has 2 rows and 2 columns, the second has 2 rows but 3 columns. The numbers n and m may be equal, or n be be larger than m, or vice versa. Matrices having the same number or rows as columns ($n = m$) are sometimes called *square matrices*.

Vectors can be considered as special matrices, having only 1 column ($m = 1$), like:

$$\begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}$$

We sometimes call them column vectors, to distinguish them from row vectors, like:

$$\begin{pmatrix} 1 & 1 & 6 & 8 \end{pmatrix}$$

But some people may refer to both column and row vectors as vectors. In these notes, when we say "vector" we mean a "column vector".

We typically use upper case letters for matrices, like A, B etc, and lower case letters for vectors like a, b, x, y etc

There are **three basic operations** that can be performed on matrices:

1. **Multiplication by a number (or scalar)**: For a given matrix A, and a given number (scalar) α, we let αA be a matrix whose entries are obtained from those of A by simply multiplying each entry of A, by the number α. For instance, when

   $$\alpha = 2 \text{ and } A = \begin{pmatrix} 1 & 2 & 6 \\ -1 & 0 & 0 \end{pmatrix},$$

   we have that

   $$\alpha A = \begin{pmatrix} 2 & 4 & 12 \\ -2 & 0 & 0 \end{pmatrix}$$

   Note that we can define the multiplication of a vector with a scalar in the same way. For instance, if

   $$\alpha = 3 \text{ and } x = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix},$$

   then

   $$\alpha x = \begin{pmatrix} 0 \\ 6 \\ -6 \end{pmatrix}$$

2. **Addition of matrices**: For two given matrices A, and B, we can define their sum, provided that they both have the same number of rows (same n), and the same number of columns (same m). In this case, if we denote the resulting matrix by $A+B$, we have that the entries of the matrix $A+B$ are defined by taking the sum of the corresponding entries in the matrices A and B. For instance, if

$$A = \begin{pmatrix} 1 & 2 & 6 \\ -1 & 0 & 0 \end{pmatrix}, \text{ and } B = \begin{pmatrix} 0 & 1 & 1 \\ -3 & 2 & -4 \end{pmatrix}$$

then the sum of A and B exists because both matrices have 2 rows, and 3 columns, and the sum equals

$$A + B = \begin{pmatrix} 1 & 3 & 7 \\ -4 & 2 & -4 \end{pmatrix}$$

Note that we can also add vectors, provided that they have the same number of entries (or equivalently, if they have the same number of rows).

3. **Multiplication of matrices**: Multiplication of matrices is the only slightly more complicated operation of the three. Given two matrices A and B, we can only define their product AB, if A is an $n \times m$ matrix and B is an $m \times p$ matrix. In this case, the product AB will be an $n \times p$ matrix. One important thing to note here is that **the number of columns of the matrix A, must equal the number of rows of the matrix B** (both equal m here). Matrices that do not satisfy this requirement, cannot be multiplied. Suppose for example that:

$$A = \begin{pmatrix} 1 & 2 & 6 \\ -1 & 0 & 0 \end{pmatrix}, \text{ and } B = \begin{pmatrix} 0 & 1 \\ -3 & 2 \\ 1 & 4 \end{pmatrix}$$

Since A is a 2×3 matrix, and B is a 3×2 matrix, they can be multiplied. We also know from what was said above that their product must be a 2×2 matrix (because $n = 2$ and $p = 2$ here). All we need to do is to define the actual product AB of A and B. Let's do it for this particular example, from which the general rule will become obvious more easily. We have that the product of A and B equals:

$$AB = \begin{pmatrix} 0 & 29 \\ 0 & -1 \end{pmatrix}$$

So how did we compute these entries? We already know that AB is a 2×2 matrix, which means that we must compute 4 entries. Let's start with the first entry of AB, that is the entry on the 1st row and 1st column. To compute this entry, we only need the 1st *row* of the matrix A, and the 1st *column* of the matrix B. Then the first entry of their product, is obtained by taking the product of each of the corresponding entries of this row vector and column vector, and then adding those:

$$\text{entry on } 1^{st} \text{ row and } 1^{st} \text{ column of } AB = (1)(0) + (2)(-3) + (6)(1) = 0.$$

To find the other entries, we proceed in similar fashion, but taking care to pick the "right" row in A and the "right" column in B:

$$\text{entry on } 1^{st} \text{ row and } 2^{nd} \text{ column of } AB = (1)(1) + (2)(2) + (6)(4) = 29.$$

entry on 2nd row and 1st column of $AB = (-1)(0) + (0)(-3) + (0)(1) = 0$.

entry on 2nd row and 2nd column of $AB = (-1)(1) + (0)(2) + (0)(4) = -1$.

Exercise: Suppose that

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 5 \end{pmatrix}, \; B = \begin{pmatrix} 2 & 3 & 4 \\ -1 & -1 & 0 \end{pmatrix}$$

Is AB defined? Is BA defined? In each case, compute the product if it is defined.

Solution: AB is defined and equals:

$$AB = \begin{pmatrix} 3 & 4 & 4 \\ -1 & 1 & 8 \end{pmatrix}$$

On the other hand, BA is is not defined because B has 3 columns, but A has only 2 rows.

This example reveals an important feature of matrix multiplication: Matrix multiplication is **not commutative**, that is, the product AB and the product BA, is not equal. In fact, in this example one of the products, namely AB, is defined, whereas the other product BA is not defined. Even in examples where AB and BA are both defined, it is often not true that they are equal (can you come up with examples?). This lack of commutativity of matrix multiplication is very different from what happens in the case of multiplication of real numbers, which is commutative. Fortunately, other properties of multiplication of numbers carry over to matrices. For example, matrix multiplication is **associative**. This means that for given matrices A, B and C, there holds that:

$$A(BC) = (AB)C,$$

provided that A is an $n \times m$, B an $m \times p$ and C an $p \times q$ matrix which guarantees that the matrix multiplications above are well-defined. The expression above is therefore sometimes written simply as ABC, because no matter where we put the (), the result is the same. Finally, the following familiar distributivity property holds for matrices as well:

$$A(B + C) = AB + AC,$$

where A is an $n \times m$, and B and C are both $m \times p$ matrices.

We now return to our motivating example at the beginning of this section. Recall that we had obtained a model there that tracks the number of juveniles $J(t)$ and number of adults $A(t)$ at time t. That model was as follows:

$$J(t+1) = 0.5A(t) \tag{24}$$

$$A(t+1) = 0.5J(t) + A(t) \tag{25}$$

We will use vectors and matrices, to represent this model in a more compact way. First, we defined a vector

$$x(t) = \begin{pmatrix} J(t) \\ A(t) \end{pmatrix}$$

This vector is a 2×1 "matrix", whose entries capture the composition of the population at time t. In a similar vain, the vector $x(t+1)$ captures the composition of the population at time $t+1$.We now define one more matrix:

$$P = \begin{pmatrix} 0 & 0.5 \\ 0.5 & 1 \end{pmatrix}$$

The key step is that the above model (24) − (25), is equivalent to the following **matrix population model**:

$$x(t+1) = Px(t) \tag{26}$$

This compact form shows that the composition of the population at time $t+1$, $x(t+1)$, can be computed as **the product of the matrix** P **and** the vector whose entries capture the composition of the population at time t, i.e. **the vector** $x(t)$. Let us use this more compact form recursively to compute the vector $x(t)$, given the initial composition vector $x(0)$:

$$\begin{aligned} x(1) &= Px(0) \\ x(2) &= Px(1) = PPx(0) \\ x(3) &= Px(2) = PPPx(0) \\ &\dots \\ x(n) &= P \dots Px(0), \end{aligned}$$

where in the last equation $P \dots P$ is the n-fold product of P with itself. We shall denote this more compactly as:

$$x(n) = P^n x(0),$$

and refer to P^n as the nth power of the matrix P. The importance of the last formula, is that it tells us how to compute the composition of the population after the nth reproductive season, $x(n)$, based only on the matrix P^n, and the initial composition vector $x(0)$. This looks very nice, but let's see if we can effectively compute this. First, we consider various powers of P:

$$\begin{aligned} P^2 &= PP = \begin{pmatrix} 0.25 & 0.5 \\ 0.5 & 1.25 \end{pmatrix} \\ P^3 &= P^2 P = \begin{pmatrix} 0.25 & 0.5 \\ 0.5 & 1.25 \end{pmatrix} \begin{pmatrix} 0 & 0.5 \\ 0.5 & 1 \end{pmatrix} = \begin{pmatrix} 0.25 & 0.625 \\ 0.625 & 1.5 \end{pmatrix} \end{aligned}$$

Let's interrupt this calculation here because it does not seem that we can distinguish a pattern that would allow us to find a nice expression for the entries of the matrix P^n, when n is large. We will show that new concepts from linear algebra will enable us to come up with a very nice formula for P^n, where n is an arbitrary positive integer. That is the final goal of these lecture goals.

Exercise: Modify the juvenile-adult population model if instead it is assumed that the survival probability of an adult is only 75%, that the probability of survival

of a juvenile to the next reproduction season is 60%, and that the expected number of offspring of an adult is 0.7. Write the model as a matrix population model

$$x(t+1) = Px(t),$$

where

$$x(t) = \begin{pmatrix} J(t) \\ A(t) \end{pmatrix},$$

and find the matrix P. Assume that

$$x(0) = \begin{pmatrix} 0 \\ 10 \end{pmatrix},$$

i.e. the initial population consists of 10 adults, but no juveniles. Use matrix multiplications to find $x(1)$, $x(2)$ and $x(3)$, the population composition at times $t = 1$, $t = 2$ and $t = 3$.

lect 24, 11/22

Systems of linear equations represented in matrix form A very common problem is to find solutions of **systems of linear equations**. These consist of n linear equations in m unknowns $x_1, x_2, \ldots, x_m$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m &= 0 \\ &\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m &= 0 \end{aligned}$$

We can rewrite this more compactly as

$$Ax = 0,$$

if we define the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2m} \\ \vdots & \vdots & \ldots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nm} \end{pmatrix},$$

and the vector containing the unknowns

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$$

It is clear that $x = 0$ (the last 0 represents a vector with m entries that are all equal to zero) is always a solution of this system. Indeed $A0 = 0$, but it is possible that the are **nonzero solutions** as well. This means that sometimes, there are vectors x which are not equal to the zero vector, that are such that $Ax = 0$. For instance:

$$\begin{aligned} x_1 + x_2 &= 0 \\ 2x_1 + 2x_2 &= 0 \end{aligned}$$

is a linear system of 2 equations in 2 unknowns, but the second equation is obtained from the first by multiplying the first equation by 2. In other words, one of the 2 equations is redundant, and so it suffices to find solutions of just 1 equation, say of:

$$x_1 + x_2 = 0.$$

Obviously, there are many solutions to this equation which are not zero, e.g. $x_1 = 1$ and $x_2 = -1$ always works, and so does $x_1 = 3$ and $x_2 = -3$. A moment of reflection about the geometric interpretation of the equation $x_1 + x_2 = 0$ in the (x_1, x_2)-plane, tells us that it represents a straight line through the origin. Every point on this line represents a vector with 2 entries that forms a solution of our system of 2 equations.

This raises **the following important question**. Suppose we are given a system of equations represented in matrix form:

$$Ax = 0$$

where A is an $n \times m$ matrix, **when does this system have nonzero solutions?**

We will only give an answer to this question for a very special case[6], when A is a 2×2 matrix, which is all we need for our purposes. Thus, suppose that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then the system $Ax = 0$, of 2 linear equations in the 2 unknowns x_1 and x_2 has nonzero solutions if and only if

$$ad - bc = 0.$$

Alternatively, the linear system has only the zero solution if and only if $ad - bc \neq 0$.

Let us verify this statement on the linear system above:

$$\begin{aligned} x_1 + x_2 &= 0 \\ 2x_1 + 2x_2 &= 0 \end{aligned}$$

which in matrix form is:

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We already know that it has nonzero solutions, but here we check the condition just mentioned:

$$ad - bc = (1)(2) - (1)(2) = 0$$

Here are a few more examples of linear systems that have nonzero solutions (we only state the corresponding matrix A of the system):

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 3 \end{pmatrix}, A = \begin{pmatrix} 0 & 2 \\ 0 & 3 \end{pmatrix}, A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

In all 3 cases, the condition that $ad - bc = 0$ is easily verified, and thus we know that these systems all have nonzero solutions. What are these solutions?

In general, it is instructive to **think geometrically** about systems of 2 equations in 2 unknowns as representing specific points in the (x_1, x_2)-plane that we are trying to locate. The location of these points is constrained because the coordinates of these points must satisfy both equations. Typically, an expression of the form $a_1x_1 + a_2x_2 = 0$ represents a line through the origin in this plane, unless $a_1 = a_2 = 0$, in which case *every* point in the plane is represented. So if we specify 2 such equations, we are either representing 2 lines through the origin, or a line and the entire plane, or just the plane itself (this last case happens when *both* equations have the form $0x_1 + 0x_2 = 0$). Solutions of the system are pairs (x_1, x_2) (or vectors) that satisfy

[6] In a course on linear algebra, you will learn the answer to this question in general.

both equations. If each equation represents a line through the origin, then either these lines only intersect at the origin, or *they coincide*. What is interesting about our condition $ad - bc = 0$ is that it captures all scenarios where there are nonzero solutions; equivalently, the condition $ad - bc \neq 0$ captures the case that there is only the zero solution, i.e. the origin of the plane. Let's investigate the above examples from this geometric perspective. For

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 3 \end{pmatrix}$$

the system of 2 equations is

$$\begin{aligned} 0x_1 + 0x_2 &= 0 \\ x_1 + 3x_2 &= 0 \end{aligned}$$

The first equation is satisfied for all pairs (x_1, x_2), i.e. for any point of the plane. The second equation clearly represents a line through the origin. Solutions to our system are therefore those points in the plane, that also lie on this line. Thus, any point on the line is a solution of the system, and clearly many of them are nonzero, as expected. For the second example,

$$A = \begin{pmatrix} 0 & 2 \\ 0 & 3 \end{pmatrix},$$

we have:

$$\begin{aligned} 2x_2 &= 0 \\ 3x_2 &= 0 \end{aligned}$$

Thinking geometrically, we see that both equations represent the *same* line $x_2 = 0$, which is the horizontal line through the origin. Can you geometrically interpret the meaning of the 3rd example? And what about the example we treated earlier, where

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}?$$

lect 25, 11/27

Eigenvalues and eigenvectors of square matrices We now turn to a very important pair of concepts in linear algebra, of eigenvalues and eigenvectors.

Definition: We say that the pair (λ, x), where λ is a scalar (i.e., a number), and x is a vector, is an (eigenvalue,eigenvector)-pair of a given $n \times n$ matrix A, if x is a nonzero vector, and if:

$$Ax = \lambda x \tag{27}$$

Given an $n \times n$ matrix, how do we find these (eigenvalue,eigenvector pairs)? We will only address this question in the case that A is a 2×2 matrix[7].

To answer this question, we will proceed in **two steps**: First we show how to find all eigenvalues λ of a given 2×2 matrix. In a second step, we will find eigenvectors associated to each of these eigenvalues. Assume that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

1. To find the eigenvalues of A, we first rewrite the system in (27) more explicitly:

$$\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{28}$$

   Since eigenvalues are values for λ such that the above system has nonzero solutions, we know that the following must hold, from the discussion in the previous section:

$$(a-\lambda)(d-\lambda) - bc = 0 \tag{29}$$

   A moment of reflection shows that this a a quadratic equation in the unknown λ. Indeed, a, b, c and d are given numbers, so the only unknown here is λ. A quadratic equation has at most 2 solutions (more precisely, it can have two, one, or no real solutions, depending on the value of the discriminant of this quadratic equation). However, in this course, we will always encounter situations where this equation has 2 real solutions. These solutions are exactly the eigenvalues of the matrix A. Equation (29) is known as **the characteristic equation of the matrix** A.

   Example: We return to the motivating example at the beginning of this section, and wish to find the eigenvalues of the matrix

$$P = \begin{pmatrix} 0 & 0.5 \\ 0.5 & 1 \end{pmatrix}$$

   The characteristic equation of P is:

$$(0-\lambda)(1-\lambda) - (0.5)(0.5) = \lambda^2 - \lambda - \frac{1}{4} = 0,$$

   which has two solutions

$$\lambda_1 = \frac{1+\sqrt{2}}{2} \text{ and } \lambda_2 = \frac{1-\sqrt{2}}{2}$$

   These λ_1 and λ_2 are therefore the eigenvalues of the matrix P.

[7] To learn to calculate (eigenvalue,eigenvector)-pairs for general $n \times n$ matrices, you should take a course in Linear Algebra.

2. Once we have found the eigenvalues of a matrix, we still need to find associated eigenvectors. For each eigenvalue λ of A, an associated eigenvector is simply a nonzero solution of the system of linear equations (28). Note that such nonzero solutions must indeed exist, by the criterion we discussed in the previous section.

   Example: Let's find an eigenvector associated to the eigenvalue λ_1 for the matrix P. We plug the value of λ_1, in the equation (28):

$$\begin{pmatrix} 0-\lambda_1 & 0.5 \\ 0.5 & 1-\lambda_1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{1+\sqrt{2}}{2} & 0.5 \\ 0.5 & 1-\frac{1+\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{1+\sqrt{2}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1-\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

   Notice that

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix},$$

   is a nonzero solution of the system. In other words, the vector above, is an eigenvector associated to the eigenvalue λ_1. We can find an eigenvector associated to λ_2 in a similar way:

$$\begin{pmatrix} 0-\lambda_2 & 0.5 \\ 0.5 & 1-\lambda_2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{1-\sqrt{2}}{2} & 0.5 \\ 0.5 & 1-\frac{1-\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{1-\sqrt{2}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1+\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

   Notice that

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix},$$

   is a nonzero solution of the system, and thus this vector is an eigenvector associated to λ_2.

We summarize our example: The matrix

$$P = \begin{pmatrix} 0 & 0.5 \\ 0.5 & 1 \end{pmatrix}$$

has two (eigenvalue,eigenvector)-pairs, namely:

$$\left(\frac{1+\sqrt{2}}{2}, \begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix}\right) \text{ and } \left(\frac{1-\sqrt{2}}{2}, \begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix}\right)$$

Exercise Find the (eigenvalue,eigenvector)-pairs for the following 2 matrices:

$$\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 8 \\ 2 & 1 \end{pmatrix}$$

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Now that we know how to find (eigenvalue,eigenvector)-pairs for a given 2×2 matrix, we investigate what use these pairs may have. To focus the discussion, suppose that A is a given 2×2 matrix, and assume that (λ_1, v_1) and (λ_2, v_2) are (eigenvalue,eigenvector)-pairs of A, with $\lambda_1 \neq \lambda_2$. In particular, (27) must hold twice:

$$Av_1 = \lambda_1 v_1, \text{ and } Av_2 = \lambda_2 v_2 \tag{30}$$

If we define two new 2×2 matrices[8]:

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \text{ and } T = (v_1 \; v_2) \tag{31}$$

then we can rewrite (30) more compactly as:

$$AT = T\Lambda \tag{32}$$

Indeed, the latter means:

$$A\,(v_1 \; v_2) = (v_1 \; v_2) \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

which reduces to (30) when carefully performing the matrix multiplications.

We introduce one more concept, known as an **invertible matrix**:

Definition: Let T be a given 2×2 matrix. We say that T is invertible, if there exists another 2×2 matrix U, called the inverse of T, such that

$$TU = UT = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{33}$$

This definition is reminiscent of the invertibility of real numbers: We say that a real number t is invertible if there exists another real number u such that $tu = ut = 1$. Note that the matrix in the right hand side of (33) plays the role of the unit 1 in the case of numbers. We call it the 2×2 **identity matrix and shall denote it as** I. Note that if we multiply *any* 2×2 matrix X with I (either on the left, or on the right), we obtain X again: $XI = IX = X$. This is similar to what happens for the multiplication of numbers with the unit 1: $1.x = x.1 = x$.

Of course, in the case of real numbers, we know that t is invertible if and only if $t \neq 0$, and moreover the inverse u of t is unique and equal to $1/t$. Does a similar statement hold in the case of matrices? The answer is yes, but it is a bit more subtle than in the case of numbers. First, given a 2×2 matrix T of the form:

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

it turns out that T is invertible if and only if:

$$ad - bc \neq 0, \tag{34}$$

[8]For obvious reasons, Λ is called a **diagonal** matrix.

a condition we have encountered before when we were investigating whether a system of linear equations only had the zero vector as its solution. Moreover, when T is invertible (i.e. when (34) holds), then I claim that the following matrix U is the unique inverse of T:

$$U = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{35}$$

To show that this matrix U satisfies (33), we calculate TU and UT:

$$UT = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{ad - bc} \begin{pmatrix} da - bc & db - bd \\ -ca + ac & -cb + ad \end{pmatrix} = I,$$

as desired. Show in a similar fashion that $TU = I$, which confirms that this matrix U satisfies the condition (33) in the Definition above. We will not show that this U is in fact, the *unique* matrix that satisfies (33), although this is not that hard to do (take a Linear Algebra course for this proof).

Instead of writing U for the unique inverse of a matrix T, we shall use the notation $U = T^{-1}$, in analogy to the reciprocal of a nonzero real number t being denoted as t^{-1}.

Example: Suppose that:

$$T = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Is T invertible? If so, calculate its inverse.

Answer: We first check the condition for invertibility (34):

$$(1)(4) - (2)(3) = 4 - 6 = -2 \neq 0,$$

which shows that T is invertible. To find the inverse T^{-1}, we simply use formula (35):

$$T^{-1} = \frac{1}{-2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$$

Now that we've learned when a matrix is invertible, and how to compute its inverse, we return to (32).

Fact from Linear Algebra (no proof): If the matrix 2×2 matrix A has two distinct eigenvalues λ_1 and λ_2, then the matrix T whose columns are corresponding eigenvectors v_1 and v_2, is an invertible matrix. Suppose that we multiply both sides of (32) with the inverse T^{-1} on the right. Then we get:

$$ATT^{-1} = A(TT^{-1}) = AI = A = T\Lambda T^{-1},$$

i.e.

$$A = T\Lambda T^{-1}, \tag{36}$$

which is a very important and practically useful formula as we will see in the next lecture.

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Suppose we want to calculate A^n, where n is an arbitrary positive integer. Using formula (36), we get:

$$A^n = (T\Lambda T^{-1})(T\Lambda T^{-1})\dots(T\Lambda T^{-1}),$$

where there are n factors $(T\Lambda T^{-1})$. Then by associativity of matrix multiplication, and using the fact that $TT^{-1} = I$, we find that:

$$A^n = T\Lambda(T^{-1}T)\Lambda(TT^{-1})\dots(TT^{-1})\Lambda T^{-1} = T\Lambda I\Lambda I\dots I\Lambda T^{-1} = T\Lambda^n T^{-1},$$

In other words, we have shown that (36) implies that for any positive integer n, there holds that:

$$A^n = T\Lambda^n T^{-1}. \tag{37}$$

Let's compute Λ^n. Recall that Λ is a diagonal matrix:

$$\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \Rightarrow \Lambda^n = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix},$$

that is, Λ^n is also a diagonal matrix, obtained from Λ by simply raising the diagonal entries λ_1 and λ_2 to the nth power.

Summary: Formula (37) shows that we can write a **closed-form formula** for *any* positive power of a given matrix A. This formula requires that we know the distinct eigenvalues λ_1 and λ_2 of the matrix A (to form the matrix Λ, and hence Λ^n given above), as well as the corresponding eigenvectors v_1 and v_2 (which are exactly the 2 columns of the matrix T). Once we know T, we can easily compute the inverse T^{-1} as outlined earlier, and therefore, we can evaluate the product $T\Lambda^n T^{-1}$ by a simple product of 3 known matrices.

Application: Recall from the derivation on p. 49 that the solution of the matrix population model

$$x(t+1) = Px(t)$$

was given by the formula:

$$x(n) = P^n x(0).$$

We have given a procedure that enables us to calculate a closed-form for P^n for any positive n, provided we know the (eigenvalues, eigenvector) pairs of the matrix P. But we have already determined these, see p.55:

$$\lambda_1 = \frac{1+\sqrt{2}}{2} \text{ and } \lambda_2 = \frac{1-\sqrt{2}}{2},$$

and

$$v_1 = \begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix}.$$

There follows that

$$T = \begin{pmatrix} 1 & 1 \\ 1+\sqrt{2} & 1-\sqrt{2} \end{pmatrix}, \text{ and therefore } T^{-1} = \frac{1}{-2\sqrt{2}} \begin{pmatrix} 1-\sqrt{2} & -1 \\ -(1+\sqrt{2}) & 1 \end{pmatrix},$$

and also that for any positive n:

$$\Lambda^n = \begin{pmatrix} \left(\frac{1+\sqrt{2}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{2}}{2}\right)^n \end{pmatrix}$$

We can now use formula (37) with P instead of A, to get that:

$$\begin{aligned}
x(n) &= P^n x(0) \\
&= T\Lambda^n T^{-1} x(0) \\
&= \begin{pmatrix} 1 & 1 \\ 1+\sqrt{2} & 1-\sqrt{2} \end{pmatrix} \begin{pmatrix} \left(\frac{1+\sqrt{2}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{2}}{2}\right)^n \end{pmatrix} \frac{1}{-2\sqrt{2}} \begin{pmatrix} 1-\sqrt{2} & -1 \\ -(1+\sqrt{2}) & 1 \end{pmatrix} x(0) \\
&= \begin{pmatrix} 1 & 1 \\ 1+\sqrt{2} & 1-\sqrt{2} \end{pmatrix} \begin{pmatrix} -\frac{1-\sqrt{2}}{2\sqrt{2}} \left(\frac{1+\sqrt{2}}{2}\right)^n & \frac{1}{2\sqrt{2}} \left(\frac{1+\sqrt{2}}{2}\right)^n \\ \frac{1+\sqrt{2}}{2\sqrt{2}} \left(\frac{1-\sqrt{2}}{2}\right)^n & -\frac{1}{2\sqrt{2}} \left(\frac{1-\sqrt{2}}{2}\right)^n \end{pmatrix} x(0) \\
&= \frac{1}{2\sqrt{2}} \begin{pmatrix} -(1-\sqrt{2}) \left(\frac{1+\sqrt{2}}{2}\right)^n + (1+\sqrt{2}) \left(\frac{1-\sqrt{2}}{2}\right)^n & \left(\frac{1+\sqrt{2}}{2}\right)^n - \left(\frac{1-\sqrt{2}}{2}\right)^n \\ \left(\frac{1+\sqrt{2}}{2}\right)^n - \left(\frac{1-\sqrt{2}}{2}\right)^n & (1+\sqrt{2}) \left(\frac{1+\sqrt{2}}{2}\right)^n - (1-\sqrt{2}) \left(\frac{1-\sqrt{2}}{2}\right)^n \end{pmatrix} x(0)
\end{aligned}$$

Thus, if the initial population consists of 1 juvenile and no adults (i.e. $x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$), then the population composition at any positive time n is:

$$x(n) = \frac{1}{2\sqrt{2}} \begin{pmatrix} -(1-\sqrt{2}) \left(\frac{1+\sqrt{2}}{2}\right)^n + (1+\sqrt{2}) \left(\frac{1-\sqrt{2}}{2}\right)^n \\ \left(\frac{1+\sqrt{2}}{2}\right)^n - \left(\frac{1-\sqrt{2}}{2}\right)^n \end{pmatrix}.$$

Notice how powerful the above closed-form formula for $x(n)$ is: Given any initial composition vector $x(0)$, it is easy to write down the composition vector for any future time.