Second part of Excercise 12.21 (Newton-Rhapson's method)

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We've already proved in class that if I is an open interval, and if $f : I \to \mathbb{R}$ is convex and differentiable in I, then for $\xi \in I$,

$$f(x) - f(\xi) \geq f'(\xi)(x - \xi), \quad \forall x \in I. \tag{1}$$

Now suppose that $\phi : \mathbb{R} \to \mathbb{R}$ is strictly increasing, convex and differentiable and $\phi(\xi) = 0$. Let $x_1 > \xi$, and consider the iteration

$$x_{n+1} = x_n - \frac{\phi(x_n)}{\phi'(x_n)}, \quad n = 1, 2, \ldots \tag{2}$$

Then prove that the sequence $x_n \to \xi$ as $n \to \infty$.

Remark 1. Observe that ξ is the unique root of ϕ, since ϕ is strictly increasing. Also notice that this result yields an algorithm to find an *approximation* of the root ξ of the function ϕ (only an approximation, because we assume that an algorithm terminates after a finite number of steps).

Proof. We will first show that the sequence x_n is decreasing and bounded below (by ξ) and therefore converges to some l (by Theorem 4.17). Then we will show that l is in fact ξ, at which point we'll be done.

To prove the first assertion we will prove the following

If n is such that $\xi \leq x_n$, then $\xi \leq x_{n+1} \leq x_n$.

Proof of claim: If $\xi \leq x_n$, then $0 = \phi(\xi) \leq \phi(x_n)$ since ϕ is increasing. Moreover, $\phi'(x_n)$ must be positive. This is because ϕ' is positive on $\mathbb{R}$. To see this, assume on the contrary that there is some real number c where $\phi'(c) = 0$ (notice that ϕ' can not be negative since ϕ is strictly increasing, hence $\phi' \geq 0$ on $\mathbb{R}$). Then $\phi'(y)$ would also be 0 for all $y \leq c$ since ϕ' is increasing on $\mathbb{R}$ (this follows from Theorem 12.18 since ϕ is convex). But this would imply that ϕ would be constant on the interval $(-\infty, c]$ (by Theorem 11.17), which would contradict that ϕ is strictly increasing.

So we have that $\phi(x_n) \geq 0$ and $\phi'(x_n) > 0$ and then (2) implies that $x_{n+1} \leq x_n$.

To prove that $\xi \leq x_{n+1}$, consider (1) with $f = \phi$, $x = x_{n+1}$ and $\xi = x_n$. Then using (2), we find that

$$\phi(x_{n+1}) - \phi(x_n) \geq \phi'(x_n)(x_{n+1} - x_n) = -\phi(x_n),$$

and therefore that $\phi(x_{n+1}) \geq 0 = \phi(\xi)$. This implies that $x_{n+1} \geq \xi$ (since ϕ is strictly increasing), so we have established the claim.

So knowing that our claim holds, how does one actually prove the first assertion? Notice that if we can show that the "If" part of the claim holds for all n, then the first assertion follows. Well, for $n = 1$ this "If" part is obviously true since we were given that $x_1 > \xi$. Using the above claim recursively, we see that the "If" part holds for all $n = 2, 3, \ldots$.

To prove the second assertion, let l be the limit of the decreasing sequence x_n. Continuity of ϕ implies that $\phi(x_n) \to \phi(l)$ as $n \to \infty$. Also, since ϕ' is increasing and bounded below (by 0), the right limit

$$\lim_{x \to l+} \phi'(x)$$

exists (Theorem 12.4) and we denote it by $\phi'(l+)$. Note that $\phi'(l+) > 0$ (if it were 0 we could find a real number $c < l$ where $\phi'(c) = 0$, which is impossible as shown above). Then Theorem 8.9 and the remark following it show that

$$\phi'(x_n) \to \phi'(l+) \text{ as } n \to \infty.$$

Taking limits for $n \to \infty$ in (2) finally yields

$$l = l - \frac{\phi(l)}{\phi'(l+)},$$

and therefore $\phi(l) = 0$ from which follows that $l = \xi$. $\square$