Proof of Proposition 6.17 (Ratio Test)

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Proposition 1. *Let* $\sum_{n=1}^{\infty} a_n$ *be a series with*

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = l. \tag{1}$$

If $l < 1$*, then the series* $\sum_{n=1}^{\infty} a_n$ *converges. If* $l > 1$*, then the series diverges.*

Proof. *Case 1*: $l < 1$.

From (1) follows that if we choose $\epsilon > 0$ small enough such that $q := l + \epsilon < 1$, then there is some N, such that if $n > N$,

$$\frac{|a_{n+1}|}{|a_n|} < q.$$

More specifically,

$$|a_{N+2}| < q|a_{N+1}|, \;\; |a_{N+3}| < q|a_{N+2}| < q^2|a_{N+1}|, \ldots, |a_{N+m}| < q^{m-1}|a_{N+1}|, \; \forall m > 1.$$

Now, since $0 < q < 1$, it follows (geometric series!) that

$$\sum_{m=2}^{\infty} q^{m-1}|a_{N+1}| = (\frac{1}{1-q} - 1)|a_{N+1}|$$

and then the comparison test yields that $\sum_{m=2}^{\infty} a_{N+m}$ converges. This implies that the series $\sum_{n=1}^{\infty} a_n$ converges too because $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_{N+1} + \sum_{m=2}^{\infty} a_{N+m}$.

Case 2: $l > 1$. This time choose $\epsilon > 0$ small enough so that $q := l - \epsilon > 1$. Then there is some N such that if $n > N$,

$$\frac{|a_{n+1}|}{|a_n|} > q$$

In particular,

$$|a_{N+2}| > q|a_{N+1}|, \;\; |a_{N+3}| > q|a_{N+2}| > q^2|a_{N+1}|, \ldots, \; |a_{N+m}| > q^{m-1}|a_{N+1}|, \; \forall m > 1.$$

Since $q > 1$, it is clear that it is impossible that $\lim_{n\to\infty} a_n = 0$ (in fact, $\lim_{n\to\infty} |a_n| = +\infty$!), which is necessary for convergence of the series $\sum_{n=1}^{\infty} a_n$ (see Theorem 6.9).

□