Third problem on exam II*

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We assume mass action kinetics for the following chemical reaction

$$X + Y \leftrightarrow C \rightarrow Y + Z,$$

where the positive rate constants are denoted by k_{1+}, k_{1-} and k_2. Assume that X is supplied at a constant rate $\alpha > 0$ and that z degrades at rate $\beta > 0$:

$$\rightarrow X, \quad Z \rightarrow$$

1. Write down the differential equations for the concentrations x, y, c and z.

$$\begin{aligned}
\dot{x} &= -k_{1+}xy + k_{1-}c + \alpha \\
\dot{y} &= -k_{1+}xy + (k_{1-} + k_2)c \\
\dot{c} &= k_{1+}xy - (k_{1-} + k_2)c \\
\dot{z} &= k_2 c - \beta z
\end{aligned}$$

2. Suppose that the complex c satisfies the quasi-steady-state assumption:

$$\frac{dc}{dt} \equiv 0 \quad \Rightarrow \quad c = \frac{k_{1+}}{k_{1-} + k_2} xy$$

Simplify the equations accordingly and show that the system describing x, y and z is linear:

Plug previous expression into the x, y and z equation to get:

$$\begin{aligned}
\dot{x} &= -kxy + \alpha \\
\dot{y} &= 0 \\
\dot{z} &= kxy - \beta z,
\end{aligned}$$

where

$$k = \frac{k_{1+}k_2}{k_{1-} + k_2} > 0.$$

Since the middle equation implies that $y(t) = y_0$, we see that upon replacing y by the constant y_0 in the remaining equations, we get a linear system.

Assume that the initial concentrations of X, Y and Z are positive, and denote the initial concentration of Y by $y_0 > 0$. Show that the solution converges to the steady state

$$(\bar{x}, \bar{y}, \bar{z}) = (\frac{\alpha}{ky_0}, y_0, \frac{\alpha}{\beta}),$$

Convergence of $y(t)$ to y_0 is trivial (recall that $y(t) = y_0$), so the only thing left to show is that the x and z components converge. The remaining equations are:

$$\begin{aligned}
\dot{x} &= -ky_0 x + \alpha \\
\dot{z} &= ky_0 x - \beta z
\end{aligned}$$

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In matrix form:

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} -ky_0 & 0 \\ ky_0 & -\beta \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$$

By the note below, and since the matrix above has two negative eigenvalues $-ky_0$ *and* $-\beta$*, we conclude that solutions of the last system converge to* $(\alpha/ky_0, y_0, \alpha/\beta)$.

(Note: You may use the fact that if $\dot{X} = AX + B$ is a non-homogeneous linear system of arbitrary dimension n which is such that all eigenvalues of A are in the open left half plane, then all solutions converge to the steady state $\bar{X} = -A^{-1}B$.)