Homework assignment 3*

February 7, 2006

1. Using the method of characteristics, solve the following equation:
$$\frac{\partial x}{\partial a} + \mathrm{e}^{-t}\,\frac{\partial x}{\partial t} = -dx.$$
Here d is a positive constant and a and t are interpreted as age and time. The initial and boundary conditions (both are supposedly known) are:
$$x(a,0) = x_0(a), \;\; x(0,t) = b(t)$$

2. This problem explores some properties of a *discrete-time* random walk on the integers $\mathbb{Z}$.

   Let λ be the probability to move right and $\mu = 1 - \lambda$ the probability to move left (so a move must be made at each instant of time). Let $p_n(t)$ be the probability that at time t the position of the random walker is n. Assume that the random walk starts at $n = 0$, so that $p_0(0) = 1$ and $p_i(0) = 0$ for $i \neq 0$. First show that
$$p_n(t+1) = \lambda p_{n-1}(t) + \mu p_{n+1}(t), \;\; n \in \mathbb{Z}.$$
Second, using the method of the generating function, determine the mean $m(t)$ and variance $\sigma^2(t)$ of the position of the random walker at time t. Verify the plausibility of your results by considering the case $\lambda = \mu = 0.5$.

3. In class we showed how to solve the renewal equation:
$$b(t) = \int_0^t b(t-a)l(a)m(a)da + \int_0^\infty u_0(r)\frac{l(r+t)}{l(r)}m(t+r)dr.$$
Unfortunately our method was based on a contraction mapping argument which only shows that a unique solution $b(t)$ *exists*. Here we consider the following special case:
$$d(a) = d > 0, \;\; m(a) = m > 0, \text{ and } U_0 := \int_0^\infty u_0(a)da.$$
In other words, death rate and maturity function are assumed to be constant and the total initial population is U_0. Determine $b(t)$ explicitely and calculate $\lim_{t\to\infty} b(t)$ (Hint: Use Laplace transforms to calculate $b(t)$). How do the values of the parameters affect the value of this limit? Calculate R_0 and discuss how the parameters affect it. Explain your findings.

4. The characteristic equation associated to McKendrick's equation is:
$$F(\lambda) = 1,$$
where $F(\lambda) := \int_0^\infty f(a)\,\mathrm{e}^{-\lambda a}\,da$ is the Laplace transform of the net maternity function $f(a) = l(a)m(a)$. Assume that λ is a real variable and that f is zero outside $[\alpha,\beta]$ ($\alpha < \beta$ are both positive) and positive and continuous on $[\alpha,\beta]$. Prove that
   - F is continuous and strictly decreasing (ie $\lambda_1 < \lambda_2$ implies that $F(\lambda_2) < F(\lambda_1)$).
   - $\lim_{\lambda\to-\infty} F(\lambda) = +\infty$ and $\lim_{\lambda\to+\infty} F(\lambda) = 0$.
   - (Optional: For people with a background in complex variables) Let λ^* be the unique real root of the characteristic equation. Now assume that λ is a complex variable. Show that λ^* is dominant in the sense that every other (complex) root of the characteristic equation has a real part strictly less than λ^*.

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