Homework assignment 3*

Due date: Monday April 23, 2007.

1. Find the matrix exponentials of the following matrices (Don't use software for your calculations! You can use only use it to verify your results.):

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

2. Give an example of two 2×2 matrices A and B such that $\mathrm{e}^{A+B} \neq \mathrm{e}^A \mathrm{e}^B$.

3. Determine stability of the zero solution of the following equations:

   - $\ddot{x} + \dot{x} + \sin x = 0$.
   - $\ddot{x} + \dot{x} \cos x + \sin x = 0$.

   Note that these are problems 12.5 # 13 and # 14 from our text. Ignore the hint given there, and instead try the following function:

$$V(x,y) = 2\sin^2\left(\frac{x}{2}\right) + \frac{y^2}{2}.$$

4. Show that the competition model

$$\begin{aligned} \dot{x} &= x(-ax - by + r_1) \\ \dot{y} &= y(-cx - dy + r_2), \end{aligned}$$

   where all parameters a, b, c, d, r_1 and r_2 are positive, does not have a non-constant periodic solution in the region D:

$$D = \{(x,y) \mid x > 0, y > 0\}.$$

   Hint: Apply Dulac's criterion with appropriately chosen positive function $B(x,y)$.

5. A system is called *Hamiltonian* if there is some differentiable function $H : \mathbb{R}^2 \to \mathbb{R}$ such that the equations of the system are given by:

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial y}(x,y) \\ \dot{y} &= -\frac{\partial H}{\partial x}(x,y) \end{aligned}$$

   Suppose that $(0,0)$ is a critical point of the function H (i.e. $\partial H/\partial x(0,0) = \partial H/\partial y(0,0) = 0$), and that H is positive definite in some open neighborhood U of $(0,0)$.

   Show that $(0,0)$ is a stable equilibrium of the corresponding Hamiltonian system. Can $(0,0)$ be asymptotically stable?

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