Homework assignment 2*

Due date: Monday February 25, 2008.

1. (# 3.2.2 (g)) Sketch $f(x)$, the Fourier series of $f(x)$ and calculate the Fourier coefficients when $f(x)$, $x \in [-L, L]$ is given by:

$$f(x) = \begin{cases} 1, & x < 0 \\ 2, & x > 0 \end{cases}$$

2. (slight modification of # 3.3.2 (c)) Let $f(x)$, $x \in [0, L]$ be defined as

$$f(x) = \begin{cases} 0, & x < \frac{L}{2} \\ x, & x > \frac{L}{2} \end{cases}$$

Sketch **both** Fourier sine series **and** Fourier cosine series of $f(x)$, and calculate their coefficients.

3. (3.3.18) Let $f(x)$ be piecewise smooth and continuous. (a) Under what conditions are f and its Fourier series equal for all $x \in [-L, L]$. (b) same question for Fourier sine series but $x \in [0, L]$. (c) same question for Fourier cosine series and $x \in [0, L]$.

4. For integer $N \geq 1$ we defined the Dirichlet kernel

$$D_N(x) = \begin{cases} \frac{\sin\left((N+\frac{1}{2})x\right)}{2\pi \sin\left(\frac{x}{2}\right)} & \text{if } x \neq k2\pi, \ k \in \mathbb{Z} \\ \frac{1}{\pi}\left(N + \frac{1}{2}\right) & \text{if } x = k2\pi, \ k \in \mathbb{Z} \end{cases}$$

   - Prove that if $x \neq k2\pi$, $k \in \mathbb{Z}$, then

   $$D_N(x) = \frac{1}{\pi}\left(\frac{1}{2} + \cos x + \cos 2x + \cdots + \cos Nx\right).$$

   (Hint: Use formulas for products of trig functions.)
   - Show that $D_N(x)$ is continuous everywhere (Hint: Recall the famous limit of $\sin x/x$ when $x \to 0$).
   - **Bonus**: Use your favorite software to plot $D_N(x)$ for $N = 1, 2, \ldots, 5$, and print the sheet with the commands you used, and of course also showing the graphs of the $D_N(x)$'s.

5. Let

$$f(x) = x, \quad x \in (-\pi, \pi)$$

Using your favorite software, plot the Fourier sine series of f. Also plot the partial sum of the Fourier sine series for $N = 5$ and $N = 10$, illustrating the Gibbs phenomenon at $x = \pi$.

6. (# 4.4.7) Let a vibrating string satisfy:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(L,t) = 0, \quad u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = 0.$$

Show that

$$u(x,t) = \frac{1}{2}\left(F(x - ct) + F(x + ct)\right),$$

where F is the odd periodic extension of f.

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7. (# 4.4.9) From the wave equation
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$
describing a vibrating string, show that the total energy, defined as the sum of kinetic and potential energy:
$$E(t) = \int_0^L \frac{1}{2} \left(\frac{\partial u}{\partial t} \right)^2 dx + \int_0^L \frac{c^2}{2} \left(\frac{\partial u}{\partial x} \right)^2 dx, \tag{1}$$
satisfies
$$\frac{dE}{dt}(t) = c^2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial t} |_0^L. \tag{2}$$
Show that if the two ends of the string at $x = 0$ and $x = L$ are fixed, then the total energy remains constant in time.

8. (#4.4.12) Using (1) – (2), show that the solution to
$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = u(L,t) = 0, \quad u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x),$$
is unique. (Hint: Let u and v be two solutions. Consider $w = u - v$.)