Homework assignment 3*

Due date: Wednesday April 2, 2008.

1. (parts of # 5.8.8) Consider the BVP

$$y'' + \lambda y = 0,\ y \in (0,1),\ \ y(0) - y'(0) = y(1) + y'(1) = 0.$$

   - Use the Raleigh quotient to show that $\lambda \geq 0$. Explain why $\lambda > 0$.
   - Show that

$$\tan(\sqrt{\lambda}) = \frac{2\sqrt{\lambda}}{\lambda - 1}.$$

     Determine the eigenvalues graphically, and estimate the large eigenvalues.

2. (adaptation of # 9.3.5) Find the Green's function for

$$y'' = f,\ \ x \in (0, L),\ \ y(0) = y'(L) = 0,$$

   in 2 ways:

   - using the method of variation of parameters.
   - using the method of eigenfunction expansion.

3. Show that the nth order distributional derivative of the Dirac delta function $\delta(x)$ is given by

$$\delta^{(n)}(x)[f] = (-1)^n f^{(n)}(0),$$

   where $f \in C_c^\infty$, the space of smooth functions on $\mathbb{R}$ with compact support.

4. Let g be a smooth function on $\mathbb{R}$ and $H(x)$ be the Heaviside function.

   - Show that the distributional derivative of $g(x)H(x)$ is given by $g'(x)H(x) + g(x)\delta(x)$. (Notice that formally this is the result one expects from Leibniz's law for a product; but keep in mind that this law is not applicable here since $H(x)$ is not differentiable)
   - Show that the distributional integral $\int_{-\infty}^{x} g(y)\delta(y - x_0)dy$ is given by $g(x_0)H(x - x_0)$.

5. (parts of of # 9.3.6) Solve for the Green's function directly:

$$\frac{d^2G}{dx^2} = \delta(x - x_0),\ \ x \in (0, L),\ \ G(0, x_0) = \frac{dG}{dx}(L, x_0) = 0,$$

   and compare your result to that of problem 2 above.

6. (parts of # 9.3.11) Considering the Helmholtz equation and assuming that L is no multiple of π, solve for the Green's function directly:

$$\frac{d^2G}{dx^2} + G = \delta(x - x_0),\ \ x \in (0, L),\ \ G(0, x_0) = G(L, x_0) = 0.$$

   Show that G is symmetric: $G(x, x_0) = G(x_0, x)$. (Note: use the variation of parameters method to solve for G directly, and use the rules of distributional integration; for instance, you might find that the results of problem 4 come in handy.)

*MAP 4341; Instructor: Patrick De Leenheer.