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CLASSROOM NOTES IN APPLIED MATHEMATICS

EDITED BY MURRAY S. KLAMKIN

This section contains brief notes which are essentially self-contained applications of mathematics that can be used in the classroom. New applications are preferred, but exemplary applications not well known or readily available are accepted.

Both "modern" and "classical" applications are welcome, especially modern applications to current real world problems.

Notes should be submitted to M. S. Klamkin, Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1.

A SIMPLE PROOF THAT THE WORLD IS THREE-DIMENSIONAL*

TOM MORLEY†

Abstract. The classical Huygens' principle implies that distortionless wave propagation is possible only in odd dimensions. A little known clarifications of this principle, due to Duffin and Courant, states that radially symmetric wave propagation is possible only in dimensions one and three. This paper presents an elementary proof of this result.

1. Introduction. The title is, of course, a fraud. We prove nothing of the sort. Instead we show that radially symmetric wave propagation is possible only in dimensions one and three.

In 1864 James Clark Maxwell discovered the fundamental laws of electromagnetism; see [5]. Maxwell's theory predicted the existence of electromagnetic radiation, i.e., electromagnetic waves. It was not until 1888 that Heinrich Rudolf Hertz discovered radio waves in the laboratory. (By the way, Hertz was a student of Helmholtz.) There can be little doubt that this discovery and subsequent technological advances have had a profound effect on modern life. What would the world be like without radio, television, and global instantaneous (or nearly so) communication? It is the purpose of this note to give a short elementary proof that this state of affairs can exist only in three dimensions. In particular:

THEOREM. *Radially symmetric distortionless wave propagation is possible only in dimensions one and three. However, in one dimension there is no attenuation.*

For precise definitions of these terms see §2.

This theorem was proved by R. J. Duffin in 1952 [3], and is mentioned by R. Courant in [2]. (It is not known whether Courant knew of Duffin's work.) Neither Courant nor Duffin ever published a proof. The present proof, however, is different and considerably more elementary than Duffin's original proof, and is suitable for presentation in the typical junior-senior level ODE–PDE course.

2. Radial wave propagation. Consider the n-dimensional wave equation

$$\sum u_{x_i x_i} = \frac{1}{c^2} u_{tt}. \tag{W}$$

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†School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332.

A radially symmetric wave is a solution of (W) that depends only on t and

$$r=\left(x_1^2+x_2^2+\cdots+x_n^2\right)^{1/2}.$$

Setting $v(r,t)=u(x,t)$ we obtain, by the chain rule, the n-dimensional radially symmetric wave equation

$$\text{(RW)}\qquad v_{rr}+\frac{n-1}{r}v=\frac{1}{c^2}v_{tt}.$$

DEFINITION. Distortionless radially symmetric wave propagation is possible if there are functions $\alpha(r)>0$, $\delta(r)>0$, $\delta(0)=0$, and $\alpha(0)=1$ such that given any "reasonable" f, the function

$$\alpha(r)f(t-\delta(r))$$

is a solution of (RW). The function $\alpha(\cdot)$ is termed the attenuation, and the function $\delta(\cdot)$ is the delay. If α is identically 1 then there is no attenuation.

It should be noted that "reasonable" can be quite unrestrictive; the class of polynomials or trigonometric polynomials will suffice.

Proof of theorem. If distortionless radially symmetric wave propagation is possible, then given any reasonable f the function $v(r,t)=\alpha(r)f(t-\delta(r))$ is a solution of (RW). Computing partial derivatives:

$$\begin{aligned}
&v_{tt}=\alpha f'',\\
&v_r=\alpha' f-\alpha\delta' f',\\
&v_{rr}=\alpha'' f-\alpha'\delta' f'-(\alpha'\delta'+\alpha\delta'')f'+\alpha\delta'^2 f''.
\end{aligned}$$

Plugging these values into (RW), we obtain

$$(*)\qquad \alpha'' f-\alpha'\delta' f'-(\alpha'\delta'+\alpha\delta'')f'+\alpha\delta'^2 f''+\frac{n-1}{r}(\alpha' f-\alpha\delta' f')=\frac{\alpha}{c^2}f''.$$

In the above computations, the arguments of the functions have been deleted for notational convenience. For instance, f is an abbreviation for $f(t-\delta(r))$.

The only possible way for (*) to hold for all reasonable f is for the coefficients of f'',f' and f to each be equal to zero. Equating the coefficient of f'' to zero, gives

$$(1)\qquad \alpha\delta'^2=\frac{\alpha}{c^2}.$$

Together with $\delta>0$ and $\delta(0)=0$, we deduce that

$$(2)\qquad \delta=\frac{r}{c},\quad \delta'=\frac{1}{c},\quad \delta''=0.$$

Plugging this into (*) and then considering the coefficients of f gives

$$(3)\qquad \alpha''+\frac{n-1}{r}\alpha'=0.$$

Similarly, the f' terms give

$$(4)\qquad \frac{1}{c}\left(-2\alpha'+\frac{n-1}{r}\alpha\right)=0.$$

Solutions of (3) and (4) are of the form Kr^{β}, where K and β are constants. Plugging this guess for α into (3) and (4) gives:

$$\beta(\beta-1)+(n-1)\beta=0, \tag{3'}$$

$$-2\beta+(n-1)=0. \tag{4'}$$

Equations (3′) and (4′) only have a solution for β if $n=1$ or $n=3$. However, plugging in $n=1$ gives $\alpha(r)=1$, and thus there is no attenuation. Of course, a world without attenuation would be unbearably noisy.

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