§.7 #21 (a) $x^4 y'' + \alpha^2 y = 0$, $x = \frac{1}{t}$, $\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt}\left(-\frac{1}{x^2}\right) = -t^2\frac{dy}{dt}$

at $t = \frac{1}{x}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(-t^2\frac{dy}{dt}\right) = \frac{d}{dt}\left(-t^2\frac{dy}{dt}\right)\cdot\frac{dt}{dx}$$

$$= \left(-2t\frac{dy}{dt} - t^2\frac{d^2y}{dt^2}\right)\cdot(-t^2) = t^4\frac{d^2y}{dt^2} + 2t^3\frac{dy}{dt}$$

$\Rightarrow \frac{d^2y}{dt^2} + \frac{2}{t}\frac{dy}{dt} + \alpha^2 y = 0$, $t=0$ is reg. sing. pt.

(b) $r(r-1) + 2r = 0 \to r(r+1) = 0$, $r_1 = 0$, $r_2 = -1$, $\Delta r = +1$ positive integer

$y_1(t) = t^0\sum_{n=0}^{\infty} a_n t^n$, $a_0 \neq 0$, $y_2(t) = Cy_1\ln t + \sum_{n=0}^{\infty} b_n t^{n-1}$ $b_0 \neq 0$

$$t\left(\sum_{n=0}^{\infty} n(n-1)a_n t^{n-2}\right) + 2\sum_{n=0}^{\infty} n a_n t^{n-1} + \alpha^2 t\sum_{n=0}^{\infty} a_n t^n = 0 \to \sum_{n=1}^{\infty} n(n+1)a_n t^{n-1} + \alpha^2\sum_{n=0}^{\infty} a_n t^{n+1} = 0$$

$$\to \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+1}t^n + \alpha^2\sum_{n=1}^{\infty} a_{n-1}t^n = 0 \to 2a_1 + \sum_{n=1}^{\infty}\left[(n+1)(n+2)a_{n+1} + \alpha^2 a_{n-1}\right]t^n = 0$$

to pull off $n=0$

$a_1 = 0$ $\Rightarrow a_1 = a_3 = a_5 = \cdots = a_{2k+1} = 0$

$a_{n+1} = -\frac{\alpha^2}{(n+1)(n+2)}a_{n-1}$, $n = 1, 2, \ldots$ $a_2 = -\frac{\alpha^2}{2\cdot 3}a_0$, $a_4 = +\frac{\alpha^4}{2\cdot3\cdot4\cdot5}a_0$

$a_6 = -\frac{\alpha^6}{2\cdot3\cdot4\cdot5\cdot6\cdot7}a_0$, $a_8 = +\frac{\alpha^8}{9!}a_0$

$\Rightarrow a_{2n} = (-1)^n\frac{\alpha^{2n}}{(2n+1)!}a_0$, $n = 0, 1, \ldots$

$y_1 = a_0\left(1 - \frac{\alpha^2}{3!}t^2 + \frac{\alpha^4}{5!}t^4 - \frac{\alpha^6}{7!}t^6 + \frac{\alpha^8}{9!}t^8 - \cdots\right)$

$y_2(t) = Cy_1\ln t + \sum_{n=0}^{\infty} b_n t^{n-1}$, $b_0 \neq 0$

$y_2' = Cy_1'\ln t + C\frac{y_1}{t} + \sum_{n=0}^{\infty}(n-1)b_n t^{n-2}$

$y_2'' = Cy_1''\ln t + 2C\frac{y_1'}{t} - \frac{Cy_1}{t^2} + \sum_{n=0}^{\infty}(n-1)(n-2)b_n t^{n-3}$

$$\Rightarrow 0 + 2C\frac{y_1'}{t} - \frac{Cy_1}{t^2} + 2C\frac{y_1}{t^2} + \sum_{n=0}^{\infty}(n-1)(n-2)b_n t^{n-3} + 2\sum_{n=0}^{\infty}(n-1)b_n t^{n-3} + \alpha^2\sum_{n=0}^{\infty} b_n t^{n-1} = 0$$

$$C\left[2\frac{y_1'}{t} + \frac{y_1}{t^2}\right] + \sum_{n=0}^{\infty}(n-1)n b_n t^{n-3} + \alpha^2\sum_{n=0}^{\infty} b_n t^{n-1} = 0$$

$$C\left[2\left(-\frac{2\alpha^2}{3!} + \frac{4\alpha^4}{5!}t^2 - \frac{6\alpha^6}{7!}t^4 + \frac{8\alpha^8}{9!}t^6 - \cdots\right) + \left(\frac{1}{t^2} - \frac{\alpha^2}{3!} + \frac{\alpha^4}{5!}t^2 - \frac{\alpha^6}{7!}t^4 + \frac{\alpha^8}{9!}t^6 - \cdots\right)\right]$$

$$+\left(2b_2t^{-1} + 6b_3 + 12b_4t + 20b_5t^2 + \cdots\right) + \alpha^2\left(b_0t^{-1} + b_1 + b_2t + b_3t^2 + \cdots\right) = 0$$

coeff of $t^{-2}$: $\boxed{C = 0}$

$t^{-1}$ : $b_2 = -\frac{\alpha^2}{2}b_0$, $b_0 \neq 0$

$t^0$ : $b_3 = -\frac{\alpha^2}{6}b_1$

$t^1$ : $b_4 = -\frac{\alpha^2}{12}b_2 = +\frac{\alpha^4}{24}b_0$

$t^2$ : $b_5 = -\alpha^2 b_3 = +\alpha^4 b_1$

So $y_2(t) = \underset{\neq 0}{b_0}\left(\frac{1}{t} - \frac{\alpha^2}{2}t + \frac{\alpha^4}{24}t^3 - \cdots\right) + b_1\underbrace{\left(1 - \frac{\alpha^2}{6}t^2 + \frac{\alpha^4}{5!}\cdots\right)}_{=y_1}$

(c) In $y_2(t)$, replace $t$ by $\frac{1}{x}$.