Suggested Problems

Lyapunov - Section 10.8

# 2 \( x' = -x \cos^2 y \) \( y' = (6-x) y^2 \)

Let \( V = y - x^2 \). Then

\[
V = -2x(-x \cos^2 y) + 1 \cdot (6-x) y^2 = x^2 \cos^2 y + (6-x) y^2
\]

This is strictly positive when \( 1y1 < \frac{\pi}{2} \) and \( x < 6 \) for \( (x,y) \neq (0,0) \).

Now \( V = 0 \) on the parabola \( y = x^2 \) and is positive above this parabola.

Every neighborhood of \((0,0)\) contains points lying above the parabola, so the hypotheses of Theorem 10.5 are satisfied, and we conclude that \((0,0)\) is unstable.

# 6 \( x' = x^5(1+y^2) \) \( y' = x^2y + y^3 \)

\[
V = \frac{1}{3}(x^2 + y^2) \\
\dot{V} = x^6(1+y^2) + y(x^2y + y^3) \\
= x^6(1+y^2) + x^2y^2 + y^4 > 0
\]

Again unstable by Theorem 10.5

# 7 \( x' = x^3(1+y) \) \( y' = x^2(4+y^2) \)

\[
V = \frac{1}{2}(x^2+y^2) \\
\dot{V} = x^3(1+y) + y(x^2(4+y^2)) \\
= x^3(1+y) + y(x^2(4+y^2)) > 0 \text{ in } (x,y) \neq (0,0)
\]

Unstable, by Theorem 10.5

if also \( 1+y > 0 \) or \( y > -1 \).
Suggested Problems — periodic orbits

1. \( x' = x - y - x^3 \)
\( y' = x + y - y^3 \) Show there are concentric circles around the origin so that 4 points into the annulus between them and which contains no critical points. 

This was done in class except for the “no critical points” claim.

Solve \( 0 = x - y - x^3 \)
\( 0 = x + y - y^3 \)
Multiplying the first equation by \( x + y \) and the second by \( x - y \) and subtracting gives that at a critical point
\( x^3 (x + y) = y^3 (x - y) \)
\( x^3 + y^3 = xy^3 - x^3 y \).
In polar form this gives
\[ r^4 (\cos^3 \theta + \sin^3 \theta) - \cos \theta \sin \theta + \cos \theta \sin \theta = 0 \]
\[ r = 0 \] is a solution.
\[ 1 - \frac{1}{2} \sin^2 2\theta - \frac{1}{2} \sin 2\theta \cos 2\theta = 0 \]
\[ 1 - \frac{1}{2} \sin 2\theta (\sin 2\theta + \cos 2\theta) = 0 \]
Since \( \sin \) and \( \cos \) are bounded by 1, this could only hold if \( \sin 2\theta = \pm 1 \)
\( \cos 2\theta + \sin 2\theta = \pm 2 \).
But if \( \sin 2\theta = \pm 1 \) then \( \cos 2\theta = 0 \), so there is no solution. Thus \( r = 0 \) or \( (0,0) \) is the only critical point.

2. Section 10.9 4.18

This system is of Liouville type with \( p(x) = x^2 - 1 \) and \( q(x) = x \). These satisfy the hypotheses of Liouville’s theorem so there is a periodic orbit.
#3 10.9 #14

\[ x' = 3x + 4xy + xy^2 = F \\
 y' = -2y^2 + xy \]

\[ \text{div} F = 3 + 4y + y^2 - 4y + xy^2 = 3 + y^2 + xy^2 > 0 \]

No periodic orbit, by Bendixson’s criterion.

#4 10.9 #20

This is not of Liénard type, nor does Bendixson criterion give any conclusion.

\[ x' = y, \quad y' = -v + vy e^{-y} \]

Let \( V = x + y^2 \).

\[ \dot{V} = 2x'y + 2y(-x + y e^{-y}) = 2y^2 e^{-y} > 0. \]

No periodic orbit is possible since along any periodic orbit the \( y \) coordinate must be somewhere positive so \( \dot{V} \) would have to be positive on some part of the orbit. But if \( \gamma(t) \) is our supposed periodic orbit with period \( T \), \( \gamma(0) = \gamma(T) \) so

\[ 0 = V(\gamma(T)) - V(\gamma(0)) = \int_0^T \dot{V} dt > 0 \]

since \( \dot{V} > 0 \) and is positive for part of the orbit.

#5 10.9 #24

This is of Liénard type with \( p(x) = 5x^4 - 12x^2 \) and \( q(x) = 4x^3 \).

\( q \) is odd, \( x^4q(x) = x^4 > 0 \) for \( x \neq 0 \), \( p \) is even and \( \int_0^x p(s) ds = x^5 - 4x^3 \)

has one positive root at \( x = 2 \), is negative for \( 0 < x < 2 \) and positive and increasing for \( x > 2 \).

So Liénard’s theorem applies to say there is an asymptotically stable periodic solution.