
2. Conway, chapter 3, section 2, problem 29 acd. (You may assume (b).)

3. Using Möbius transformations and powers, construct a conformal map from the upper half disk to the upper half plane.

4. Let $\Gamma$ a circle contained inside the unit circle with center $\sigma$ and radius $\rho$. The aim of this problem is construct a Möbius transformation $T$ which takes the unit circle to itself and $\Gamma$ to a circle centered at the origin. By a rotation in $z$ we may suppose that $\sigma$ satisfies $0 \leq \sigma < 1$. By a result from class, the transformation sought must have the form $z \to \gamma \frac{z-a}{1-\overline{a}z}$ for some $a$ of modulus less than 1 and $\gamma$ with $|\gamma| = 1$. Since the $\gamma$ factor leaves all circles centered at the origin invariant, it may be omitted.

   (a) Using symmetry of $\sigma$ and $\infty$ with respect to $\Gamma$, and properties of Möbius maps, derive an equation relating $a$, $\sigma$, and the radius $R$, of the image circle which shows that $a$ must be real.

   (b) Show that $T$ must take the points 1, $\sigma$, -1 to 1, 0, -1 respectively, and so the images of $\sigma \pm \rho$ are $\pm R$.

   (c) Show from this that

   $$a = \frac{1 + \sigma^2 - \rho^2 - \sqrt{(1 - (\sigma - \rho)^2)(1 - (\sigma + \rho)^2)}}{2\sigma}.$$ 

5. Let $T$ be a Möbius transformation mapping the unit disk onto itself. Let $z(t)$ be a differentiable curve in the unit disk with non-zero tangent vector, and let $w(t) = T(z(t))$. Prove that

$$\frac{|w'(t)|}{1 - |w(t)|^2} = \frac{|z'(t)|}{1 - |z(t)|^2}.$$ 

In geometry, this would say that $T$ is an isometry with respect to the metric $\frac{|dz|}{1-|z|^2}$. Assuming that segments of the x-axis are geodesics (curves of shortest length) for this metric, show that the general geodesic is either a segment of a diameter or a segment of a circle intersecting the unit circle perpendicularly.