Section 2.3

Definitions A sequence \(\{x_n\}\) of real numbers is
- **increasing** (resp. strictly increasing) if each \(x_{n+1} \geq x_n\), (resp \(x_{n+1} > x_n\))
- **decreasing** (resp. strictly decreasing) if each \(x_{n+1} \leq x_n\), (resp \(x_{n+1} < x_n\))
- **monotone** if it is increasing or decreasing.

Monotone Convergence Theorem If \(\{x_n\}\) is increasing and bounded above, or decreasing and bounded below, it converges.

Examples, Nested Interval Property
- If \(a > 0\), \(\{a^{(1/n)}\}\) → 1.
- If \(|a| < 1\), then \(\{a^n\}\) → 0.

Definition: A sequence of sets \(\{I_n\}_{n=1}^{\infty}\) is **nested** if each \(I_n \supset I_{n+1}\).

Theorem: If \(\{I_n\}_{n=1}^{\infty}\) is a nested sequence of closed and bounded intervals, then \(E = \cap_{n=1}^{\infty} I_n \neq \emptyset\). If the lengths of the intervals → 0, then \(E\) is a single point.

Bolzano Weierstrass Theorem

**B-W Theorem:** Every bounded sequence of real numbers has a convergent subsequence.

Sec. 2.4, Cauchy Sequences

**Definition:** A sequence \(\{x_n\}_{n=1}^{\infty}\) of reals is **Cauchy** if for each \(\varepsilon > 0\), \(\exists N_0 \in \mathbb{N}\) so that if \(n \text{ and } m \geq N_0\), then \(|x_n - x_m| < \varepsilon\).

**Theorem:** A sequence of real numbers is Cauchy if and only if it converges.

**Examples**
- If \(|x_n - x_{n+1}| \leq \frac{1}{2^n}\), then \(\{x_n\}\) converges.
- There are nonconvergent sequences with \(|x_n - x_{n+1}| \to 0\)