Main Result on Orientability

**Theorem 3.26** Let $M$ be a closed connected $n$ manifold. Then:

(a) If $M$ is $R$ orientable, the map $H_n(M; R) \to H_n(M|x; R) \cong R$ is an isomorphism for all $x \in M$.

(b) If $M$ is not $R$ orientable, the map $H_n(M; R) \to H_n(M|x; R) \cong R$ is injective with image $\{ r \in R | 2r = 0 \}$ for all $x \in M$.

(c) $H_i(M; R) = 0$ for $i > n$.

Relation to $\Gamma$

**Lemma 3.27:** Let $M$ be a manifold of dimension $n$ and let $A \subset M$ be a compact subset. Then:

(a) the homomorphism $j_A : H_n(M|A; R) \to \Gamma(A)$ given by $j_A(\alpha)(x) = (x, i_A^x(\alpha))$ is an isomorphism.

(b) $H_i(M|A; R) = 0$ for $i > n$.

Generalization

**Lemma:** Let $M$ be a manifold of dimension $n$ and let $A \subset M$ be a closed subset. Then:

(a) The homomorphism $j_A : H_n(M|A; R) \to \Gamma(A)$ is a monomorphism with image $\Gamma_c(A)$.

(b) $H_i(M|A; R) = 0$ for $i > n$.

Sections

**Definition:** Sections over $A$, $\Gamma(A)$.

**Lemma:** $\Gamma(A)$ has an $R$-module structure.

**Definition:** $v : M_R \to R \mod$ units

**Definition:** $R$ orientable along $A$. 
**Orientability along A**

**Lemma:** $M$ is $R$ orientable along $A$ if and only if there is a homeomorphism $\phi : p^{-1}(A) \to A \times R$ such that $p^{-1}(A) \xrightarrow{\phi} A \times R$ commutes. In this case, $\Gamma(A)$ is isomorphic to the module of maps from $A$ to $R$.

**Lemma:** If $j_A : H_n(M|A) \to \Gamma(A)$ is defined by $j_A(\alpha)(x) = (x, i_x^A(\alpha))$, then $j_A$ is a homomorphism.

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**Corollaries to Main Result**

**Corollary:** If $A$ is connected and non compact, $H_n(M|A; R) \simeq 0$. If $M$ is connected and non compact, $H_n(M; R) \simeq 0$.

**Corollary:** If $M$ is orientable along $A$ and $A$ is compact with $k$ components, then $H_n(M|A; R) \simeq R^k$.

**Corollary:** If $M$ is compact and connected, and $R$ has the property that for any unit $u$ and $a \neq 0$, $ua = a \Rightarrow u = 1$, then

$$H_n(M; R) \simeq \begin{cases} R & \text{if } M \text{ is } R \text{ orientable} \\ 0 & \text{otherwise} \end{cases}$$