Covering Dimension

(All spaces are separable metric in this section.)

**Definition:** The covering dimension of $X$, $\dim(X)$ is $\leq n$ if every finite open cover of $X$ has a finite open refinement of order $\leq n + 1$. $\dim(X) = n$ if $\dim(X) \leq n$ and $\dim(X) \neq n - 1$.

**Definition:** The nerve of a locally finite open cover $\mathcal{U}$ is a simplicial complex $K$ that has one vertex $v_i$ for each $U_i \in \mathcal{U}$, so that \( \langle v_{i_1}, \ldots v_{i_n} \rangle \in K \) iff $U_{i_1} \cap \ldots U_{i_n} \neq \emptyset$.

Mapping to Nerves

**Definition:** Given an open cover $\mathcal{U}$ of $X$, a map $f : X \to Y$ is a $\mathcal{U}$ map if each $f(x)$ has a neighborhood $N$ such that $f^{-1}(N)$ is in some element of $\mathcal{U}$.

**Lemma:** Let $\mathcal{U}$ be a locally finite open cover of $X$ and let $K$ be the nerve of $U$. Then there is a map $f : X \to |K|$ such that $f$ is a $\mathcal{U}$ map.

**Idea:** Let $s_i(x) = d(x, X - U_i)$, $t_i(x) = \frac{s_i(x)}{\sum_i s_i(x)}$, and $f(x) = \sum_i t_i(x) \cdot v_i$.

General Position

**Definition:** Points $x_1, x_2, \ldots$ in $R^k$ are in **general position** if every subset of cardinality $j \leq k + 1$ spans a $j - 1$ simplex.

**Lemma:** Given a countable dense set $x_1, x_2, \ldots$ in $R^k$ and an $\varepsilon > 0$, there exists a countable dense set $y_1, y_2, \ldots$ in $R^k$ in general position with $d(x_i, y_i) < \varepsilon$ for all $i$.

Mapping to $R^{2n+1}$

**Lemma:** Given a finite open cover of $\mathcal{U}$ of order $n + 1$ $X$, there is an embedding $e$ of the nerve of $\mathcal{U}$ into $R^{2n+1}$ and a map $f : X \to e(|K|)$ such that $f$ is a $\mathcal{U}$ map.

**Lemma:** If $\dim(X) = n$, $\mathcal{U}$ is a finite open cover of $X$, and $H$ is an $n$-dimensional hyperplane in $R^{2n+1}$, \( \{ f : X \to R^{2n+1} | f \text{ is a $\mathcal{U}$ map and } f(X) \cap H = \emptyset \} \) is dense and open in $C(X, R^{2n+1})$. 
Embedding in $\mathbb{R}^{2n+1}$

**Definition:**

$Q_i^n = \{ x \in \mathbb{R}^n | x \text{ has exactly } i \text{ rational coordinates} \}$

$N_i^n = \{ x \in \mathbb{R}^n | x \text{ has at most } i \text{ rational coordinates} \}$

**Theorem:** If $\dim(X) = n$, there is an embedding of $X$ into $N_n^{2n+1} \subset \mathbb{R}^{2n+1}$. ($N_n^{2n+1}$ is $n$-dimensional.)

**Note:** There are also compact universal $n$-dimensional spaces in $\mathbb{I}^{2n+1}$

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Inductive, Partition Dimension

**Definition:** $\text{ind}(\emptyset) = -1$. $\text{ind}(X) \leq n$ if $X$ has a basis $\mathcal{B} = \{U_1, U_2, \ldots \}$ with $\text{ind}(\text{Bd}(U_i)) \leq n - 1$ for each $i$.

**Theorem:** $\dim(X) \leq n$ if and only if $\text{ind}(X) \leq n$

iff

for each set $\{(A_1, B_1), (A_2, B_2), \ldots (A_{n+1}, B_{n+1})\}$ of $n + 1$ pairs of closed disjoint sets in $X$, there exist partitions $S_i$ between $A_i$ and $B_i$ with $\cap_i S_i = \emptyset$.

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Mapping to spheres

**Theorem:** $\dim(X) \leq n$ if and only if for each closed subset $A$ of $X$ and for each map $f : A \rightarrow S^n$, there exists an extension $F : X \rightarrow S^n$.

**Note:** There is a definition of cohomological dimension of a space in terms of mapping to $K(\pi, n)$, a space constructed by starting with $S^n$ and inductively adding cells of higher dimension.

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References

- Dimension Theory by Hurewicz and Wallman
- Dimension Theory by Engelking