Čech Cohomology

(All spaces in this section are locally compact subsets of ANRs, equivalently locally compact separable metric.)

**Definition:** Given $A \subset X^{\text{ANR}}$, partially order the neighborhoods of $A$ by reverse inclusion. Define a directed system of abelian groups $\{ H^q(V) \}$, for neighborhoods $V$ of $A$, by defining $i_{V_1}^{V_2} : H^q(V_1) \to H^q(V_2)$, for $V_1 \leq V_2$ to be the inclusion induced homomorphism. The $q$th Čech cohomology of $A$, $\check{H}^q(A)$, is the direct limit of this system.

Induced Homomorphisms

**Definition:** Given $A \subset X^{\text{ANR}}$, $B \subset Y^{\text{ANR}}$, and $f : A \to B$, define $f^* : \check{H}^q(B) \to \check{H}^q(A)$ as follows. Extend $f$ to $F : U \to Y$, where $U$ is a nbhd of $A$. Let $W$ be any neighborhood of $B$. Then $F : F^{-1}(W) \to W$ induces $F^* : H^q(W) \to H^q(F^{-1}(W)) \to \check{H}^q(A)$.

These homomorphisms, for each $W$, induce $f^* : \check{H}^q(B) \to \check{H}^q(A)$.

**Lemma:** The definition of $f^*$ is independent of the extension $F$ and the neighborhood $U$.

Invariance

**Homotopy:** In the above setting, if $f, g : A \to B$ are homotopic, $f^* = g^*$.

**Funtoriality:** $\text{id}^* = \text{id}$, $(f \circ g)^* = g^* \circ f^*$

**Invariance:** $\check{H}^q(A)$ depends only on $A$, and not on the embedding of $A$ in $X$.

**Corollary:** If $A$ is an ANR, $\check{H}^q(A) \simeq H^q(A)$.

References

**Note:** There is an alternate definition of Čech cohomology in terms of the cohomology of nerves of covers of $A$. This depends only on $A$ by definition and is equivalent to the definition we gave.

**References:**
- Algebraic Topology by Dold
- Algebraic Topology by Greenberg and Harper