Homotopy, Extension, and Classification

Let $\theta$ be a generator of $H^N(S^N)$

**H(N):** If $f : S^N \to S^N$ is a map of degree 0, then $f$ is homotopic to a constant map.

**E(N):** Let $(X, A)$ be a CW pair, with the cells of $X - A$ of dimension $\leq N + 1$. A map $f : A \to S^N$ extends to a map $g : X \to S^N$ iff $f^*([z]) = f^*(\theta)$, $g$ can be chosen so that $g^*([z]) = [z].$

**C(N):** Let $X$ be a CW complex of dim $\leq N$. Homotopy classes of maps from $X$ to $S^N$, $[X, S^N]$, are in 1-1 correspondence with $H^N(X)$ under the correspondence $f \to f^*(\theta)$.

Method of Proof

**H(1)**

**H(N) \implies E(N)**

**E(N) \implies H(N + 1)**

**E(N) \implies C(N)**

**Corollary:** Two maps from $S^N$ to $S^N$ are homotopic if and only if they have the same degree.

Degree, Spaces of type $(G, n)$

**Definition:** Let $f : S^N \to S^N$. The degree of $f$, $d_f$, is the integer such that $f_*([z]) = d_f \cdot [z]$ where $[z]$ is a generator of $H_N(S^N)$. Equivalently, $d_f$ is the integer such that $f^*(\theta) = d_f \cdot \theta$.

**Definition:** A path connected CW complex $X$ is of type $(G, n)$ if $\pi_i(X) = 0$ for $i \neq n$ and $\pi_n(X) = G$.

**Example:** $S^1$ is of type $(Z, 1)$.

**Theorem:** For every $n$ there are spaces of type $(Z, n)$. These spaces can be constructed by starting with $S^n$ and inductively attaching cells of dimension $\geq n + 1$.

Generalizations

**Extension:**

Let $(X, A)$ be a CW pair and let $Y$ be a space of type $(Z, N)$. A map $f : A \to Y$ extends to a map $g : X \to Y$ iff $f^*([z]) \subset im(i^*)$ where $i : A \to X$ is inclusion.

**Classification:**

Let $X$ be a CW complex and let $Y$ be a space of type $(Z, N)$. Homotopy classes of maps from $X$ to $Y$, $[X, Y]$, are in 1-1 correspondence with $H^N(X)$ under the correspondence $f \to f^*(\theta)$. 