Overview

This lecture presents a summary of the metrization theorems from sections 39-42 in the text. The main new ideas are local finiteness and paracompactness.

Local Finiteness

Def. A collection of subsets of a space $X$ is locally finite if each point in $X$ has a neighborhood intersecting only finitely many of the sets.

Lemma: Let $A$ be a locally finite collection of subsets of $X$. Then:
- Any subcollection of $A$ is locally finite
- The collection of closures of elements of $A$ is locally finite
- The closure of the union of elements of $A$ is the union of the closures of elements of $A$.

Refinements of Covers

Def. A collection $B$ of subsets of $X$ is countably locally finite if it can be written as a countable union of locally finite collections.

Def. Let $\mathcal{A}$ be a collection of subsets of $X$. A collection $\mathcal{B}$ is said to refine $\mathcal{A}$ if for each $B \in \mathcal{B}$, there is an $A \in \mathcal{A}$ with $B \subset A$. The refinement is open or closed respectively if the sets in $\mathcal{B}$ are open or closed.

Lemma: If $X$ is a metrizable and $A$ is an open covering of $X$, then there is a countably locally finite open refinement $B$ of $A$ that is also a cover.

Metrization Theorems

Nagata-Smirnov Metrization Theorem:
A space $X$ is metrizable if and only if $X$ is regular and has a basis that is countably locally finite.

Def. $X$ is paracompact if every open covering has a locally finite open refinement that covers $X$.

Smirnov Metrization Theorem: A space $X$ is metrizable if and only if it is paracompact Hausdorff and locally metrizable.