Applications of the Fundamental Group

(Sections 56-57)

The Fundamental Theorem of Algebra:
A polynomial \(x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0\) with real or complex coefficients has at least one real or complex root.

Idea of Proof

- \(f : S^1 \to S^1\) given by \(f(z) = z^n\) has \(f_*\) injective.
- \(g : S^1 \to \mathbb{R}^2 - 0\) given by \(g(z) = z^n\) is not null homotopic.
- **Special Case:** \(|a_{n-1}| + \cdots + |a_1| + |a_0| < 1\)
  Show there is a root in \(B^2\)
- **General Case:**

Antipode Preserving Maps

**Def.** A map \(h : S^n \to S^m\) is said to be antipode preserving if \(h(-x) = -h(x)\) for all \(x\).

**Theorem:** An antipode preserving map from \(S^1\) to \(S^1\) is not null homotopic.

**Theorem:** There is no antipode preserving map from \(S^2\) to \(S^1\)

Borsuk Ulam Theorem

**Borsuk-Ulam theorem for \(S^2\):**
Given a map \(f : S^2 \to \mathbb{R}^2\), there is a point \(x\) of \(S^2\) such that \(f(x) = f(-x)\)

**Bisection Theorem:**
Given two bounded polygonal regions in \(\mathbb{R}^2\), there exists a single line in \(\mathbb{R}^2\) that bisects each of them.