Analysis of Methods for the Maxwell-Random Lorentz Model

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Outline

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Maxwell's Equations

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = \mathbf{0}, \text{ in } (0, T) \times \mathcal{D} \qquad \text{(Faraday)}$$

$$\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} - \nabla \times \mathbf{H} = \mathbf{0}, \text{ in } (0, T) \times \mathcal{D} \qquad \text{(Ampere)}$$

$$\nabla \cdot \mathbf{D} = \nabla \cdot \mathbf{B} = 0, \text{ in } (0, T) \times \mathcal{D} \qquad \text{(Poisson/Gauss)}$$

$$\mathbf{E}(0, \mathbf{x}) = \mathbf{E_0}; \ \mathbf{H}(0, \mathbf{x}) = \mathbf{H_0}, \text{ in } \mathcal{D} \qquad \text{(Initial)}$$

$$\mathbf{E} \times \mathbf{n} = \mathbf{0}, \text{ on } (0, T) \times \partial\mathcal{D} \qquad \text{(Boundary)}$$

$\mathbf{E} =$	Electric field vector	$\mathbf{D} =$	Electric flux density
$\mathbf{H} =$	Magnetic field vector	$\mathbf{B} =$	Magnetic flux density
$\mathbf{J} =$	Current density	$\mathbf{n} =$	Unit outward normal to $\partial\mathcal{D}$

Constitutive Laws

Maxwell's equations are completed by constitutive laws that describe the response of the medium to the electromagnetic field.

$$\mathbf{D} = \epsilon \mathbf{E} + \mathbf{P}$$
$$\mathbf{B} = \mu \mathbf{H} + \mathbf{M}$$
$$\mathbf{J} = \sigma \mathbf{E} + \mathbf{J}_s$$

$\mathbf{P} =$	Polarization	$\epsilon =$	Electric permittivity
$\mathbf{M} =$	Magnetization	$\mu =$	Magnetic permeability
$\mathbf{J}_s =$	Source Current	$\sigma =$	Electric Conductivity

where $\epsilon = \epsilon_0 \epsilon_\infty$ and $\mu = \mu_0 \mu_r$.

Lorentz Model

We employ the physical assumption that electrons behave as damped harmonic oscillators. The polarization is then defined as the average dipole moment given by the Lorentz model:

$$\ddot{\mathbf{P}} + 2\nu\dot{\mathbf{P}} + \omega_0^2\mathbf{P} = \epsilon_0\omega_p^2\mathbf{E}$$

where ω_0 is the resonant frequency, ν is a damping coefficient, and ω_p is referred to as a plasma frequency.

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Unfortunately many materials are not well modeled by a simple Lorentz polarization, and practitioners resort to linear combinations of Lorentz models, called a multi-pole Lorentz model.

Random Polarization

The multi-pole Lorentz model motivates a continuum of Lorentz mechanisms, i.e., a distribution of dielectric parameters. This is equivalent to certain fractional-time differential equation models. We define a random polarization to be a function of (at least one) dielectric parameter treated as a random variable.

Random Polarization

The multi-pole Lorentz model motivates a continuum of Lorentz mechanisms, i.e., a distribution of dielectric parameters. This is equivalent to certain fractional-time differential equation models. We define a random polarization to be a function of (at least one) dielectric parameter treated as a random variable.

The random Lorentz model is

$$\ddot{\mathcal{P}} + 2\nu\dot{\mathcal{P}} + \omega_0^2\mathcal{P} = \epsilon_0\omega_p^2 E$$

where we let the parameter ω_0^2 be a random variable with probability distribution F on the interval (a, b). The macroscopic polarization is then taken to be the expected value of the random polarization,

$$P(t, z) = \int_a^b \mathcal{P}(t, z; \omega_0^2)\, dF(\omega_0^2).$$

Maxwell-Random Lorentz system

In a polydisperse Lorentz material, we have

$$\mu_0 \frac{\partial \mathbf{H}}{\partial t} = -\nabla \times \mathbf{E} \tag{1a}$$

$$\epsilon_0 \epsilon_\infty \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{H} - \frac{\partial \mathbf{P}}{\partial t} \tag{1b}$$

$$\ddot{\mathcal{P}} + 2\nu \dot{\mathcal{P}} + \omega_0^2 \mathcal{P} = \epsilon_0 \omega_p^2 \mathbf{E} \tag{1c}$$

with

$$\mathbf{P}(t, \mathbf{x}) = \int_a^b \mathcal{P}(t, \mathbf{x}; \omega_0^2) f(\omega_0^2) d\omega_0^2.$$

2D Maxwell-Random Lorentz Transverse Electric (TE) curl equations

For simplicity in exposition and to facilitate analysis, we reduce the Maxwell-Random Lorentz model to two spatial dimensions (we make the assumption that fields do not exhibit variation in the z direction).

$$\mu_0 \frac{\partial H}{\partial t} = -\mathrm{curl}\ \mathbf{E}, \tag{2a}$$

$$\epsilon_0 \epsilon_\infty \frac{\partial \mathbf{E}}{\partial t} = \mathbf{curl}\ H - \mathbf{J}, \tag{2b}$$

$$\frac{\partial \mathcal{P}}{\partial t} = \mathcal{J} \tag{2c}$$

$$\frac{\partial \mathcal{J}}{\partial t} = -2\nu \mathcal{J} - \omega_0^2 \mathcal{P} + \epsilon_0 \omega_p^2 \mathbf{E} \tag{2d}$$

where $\mathbf{E} = (E_x, E_y)^T, \mathbf{J} = (J_x, J_y)^T$, $\mathcal{J} = (\mathcal{J}_x, \mathcal{J}_y)^T, \mathcal{P} = (\mathcal{P}_x, \mathcal{P}_y)^T$ and $H = H_z$.

Note $\mathrm{curl}\ \mathbf{U} = \frac{\partial U_y}{\partial x} - \frac{\partial U_x}{\partial y}$ and $\mathbf{curl}\ V = \left(\frac{\partial V}{\partial y}, -\frac{\partial V}{\partial x}\right)^T$.

We introduce the random Hilbert space $V_F = (L^2(\Omega) \otimes L^2(\mathcal{D}))^2$ equipped with an inner product and norm as follows

$$(\mathbf{u}, \mathbf{v})_F = \mathbb{E}[(\mathbf{u}, \mathbf{v})_2],$$
$$\|\mathbf{u}\|_F^2 = \mathbb{E}[\|\mathbf{u}\|_2^2].$$

The weak formulation of the 2D Maxwell-Random Lorentz TE system is

$$\left(\mu_0 \frac{\partial H}{\partial t}, v\right)_2 = (-\mathrm{curl}\ \mathbf{E}, v)_2 \tag{3a}$$
$$\left(\epsilon_0 \epsilon_\infty \frac{\partial \mathbf{E}}{\partial t}, \mathbf{u}\right)_2 = (\mathbf{curl}\ H, \mathbf{u})_2 - (\mathbf{J}, \mathbf{u})_2 \tag{3b}$$
$$\left(\frac{\partial \mathcal{P}}{\partial t}, \mathbf{q}\right)_F = (\mathcal{J}, \mathbf{q})_F \tag{3c}$$
$$\left(\frac{\partial \mathcal{J}}{\partial t}, \mathbf{w}\right)_F = (-2\nu\mathcal{J}, \mathbf{w})_F + \left(-\omega_0^2 \mathcal{P}, \mathbf{w}\right)_F + \left(\epsilon_0 \omega_p^2 \mathbf{E}, \mathbf{w}\right)_F . \tag{3d}$$

for $v \in L^2(\mathcal{D})$, $\mathbf{u} \in H_0(\mathrm{curl}, \mathcal{D})^2$, and $\mathbf{q}, \mathbf{w} \in V_F$.

Theorem (Stability of Maxwell-Random Lorentz)

Let $\mathcal{D} \subset \mathbb{R}^2$ and suppose that $\mathbf{E} \in C(0,T;H_0(\mathrm{curl},\mathcal{D})) \cap C^1(0,T;(L^2(\mathcal{D}))^2)$, $\mathcal{P},\mathcal{J} \in C^1(0,T;\left(L^2(\Omega)\otimes L^2(\mathcal{D})\right)^2)$, and $H(t) \in C^1(0,T;L^2(\mathcal{D}))$ are solutions of the weak formulation for the Maxwell-Random Lorentz system along with PEC boundary conditions. Then the system exhibits energy decay

$$\mathcal{E}(t) \leq \mathcal{E}(0) \quad \forall t \geq 0,$$

where the energy $\mathcal{E}(t)$ is defined as

$$\mathcal{E}(t)^2 = \left\|\sqrt{\mu_0}\; H(t)\right\|_2^2 + \left\|\sqrt{\epsilon_0\epsilon_\infty}\;\mathbf{E}(t)\right\|_2^2 + \left\|\sqrt{\frac{\omega_0^2}{\epsilon_0\omega_p^2}}\;\mathcal{P}(t)\right\|_F^2 + \left\|\frac{1}{\sqrt{\epsilon_0\omega_p^2}}\;\mathcal{J}(t)\right\|_F^2$$

where $\|u\|_F^2 = \mathbb{E}[\|u\|_2^2]$ and $\mathcal{J} := \frac{\partial\mathcal{P}}{\partial t}$.

Proof: (for 2D)

By choosing $v = H$, $\mathbf{u} = \mathbf{E}$, $\mathbf{q} = \mathcal{P}$ and $\mathbf{w} = \mathcal{J}$ in the weak form, and adding all equations into the time derivative of the definition of $\mathcal{E}^2$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d\mathcal{E}^2(t)}{dt} =& -\left(\mathrm{curl}\ \mathbf{E}, H\right)_2 + \left(H, \mathrm{curl}\ \mathbf{E}\right)_2 - \left(\mathbf{J}, \mathbf{E}\right)_2 + \left(\frac{\omega_0^2}{\epsilon_0\omega_p^2}\mathcal{J}, \mathcal{P}\right)_F \\ & - \left(\frac{2\nu}{\epsilon_0\omega_p^2}\mathcal{J}, \mathcal{J}\right)_F - \left(\frac{\omega_0^2}{\epsilon_0\omega_p^2}\mathcal{P}, \mathcal{J}\right)_F + (\mathbf{E}, \mathcal{J})_F \\ =& -\left\|\sqrt{\frac{2\nu}{\epsilon_0\omega_p^2}}\mathcal{J}\right\|_F^2\end{aligned}$$

$$\boxed{\frac{d\mathcal{E}(t)}{dt} = \frac{-1}{\mathcal{E}(t)}\left\|\sqrt{\frac{2\nu}{\epsilon_0\omega_p^2}}\mathcal{J}\right\|_F^2 \leq 0.}$$

□

Polynomial Chaos

We wish to approximate the random polarization with orthogonal polynomials of the standard random variable ξ. Let $\omega_0^2 = r\xi + m$ and $\xi \in [-1, 1]$.

Polynomial Chaos

We wish to approximate the random polarization with orthogonal polynomials of the standard random variable ξ. Let $\omega_0^2 = r\xi + m$ and $\xi \in [-1, 1]$. Suppressing the dimension of $\mathcal{P}$ and the spatial dependence, we have

$$\mathcal{P}(\xi, t) = \sum_{i=0}^{\infty} \alpha_i(t)\phi_i(\xi) \rightarrow \ddot{\mathcal{P}} + 2\nu\dot{\mathcal{P}} + (r\xi + m)\mathcal{P} = \epsilon_0\omega_p^2 E.$$

Utilizing the Triple Recursion Relation for orthogonal polynomials:

$$\xi\phi_n(\xi) = a_n\phi_{n+1}(\xi) + b_n\phi_n(\xi) + c_n\phi_{n-1}(\xi),$$

the differential equation becomes

$$\sum_{i=0}^{\infty} \left[\ddot{\alpha}_i(t) + 2\nu\dot{\alpha}_i(t) + m\alpha_i(t)\right]\phi_i + r\alpha_i(t)\left[a_i\phi_{i+1} + b_i\phi_i + c_i\phi_{i-1}\right] = \epsilon_0\omega_p^2 E\phi_0.$$

Galerkin Projection

We apply a Galerkin Projection onto the space of polynomials of degree at most p to get:

$$\ddot{\vec{\alpha}} + 2\nu\dot{\vec{\alpha}} + A\vec{\alpha} = \vec{f}$$

where $\vec{f} = \hat{e}_1\epsilon_0\omega_p^2 E$ and

$$A = rM + mI, \quad M = \begin{pmatrix} b_0 & c_1 & 0 & \cdots & 0 \\ a_0 & b_1 & c_2 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & a_{p-2} & b_{b-1} & c_p \\ 0 & \cdots & 0 & a_{p-1} & b_p \end{pmatrix}.$$

Or we can write as a first order system:

$$\dot{\vec{\alpha}} = \vec{\beta}$$

$$\dot{\vec{\beta}} = -A\vec{\alpha} - 2\nu\vec{\beta} + \vec{f}.$$

Finite Difference Time Domain (FDTD)

We now choose a space-time discretization of the Maxwell-PC Lorentz model. Note that any scheme can be used, independently of the spectral approach in random space employed here.

The Yee Scheme (FDTD)

- This gives an explicit second order accurate scheme in time and space.
- It is conditionally stable with the CFL condition

$$\frac{c_\infty \Delta t}{h} \leq \frac{1}{\sqrt{d}}$$

  where $c_\infty = 1/\sqrt{\mu_0 \epsilon_0 \epsilon_\infty}$ is the fastest wave speed in the medium, d is the spatial dimension, and h is the (uniform) spatial step.
- The Yee scheme can exhibit numerical dispersion and dissipation.

We stagger three discrete meshes in the x and y directions and two discrete meshes in time:

$$\tau_h^{E_x} := \left\{ \left(x_{\ell+\frac{1}{2}}, y_j \right) | 0 \leq \ell \leq L-1, 0 \leq j \leq J \right\}$$
$$\tau_h^{E_y} := \left\{ \left(x_{\ell}, y_{j+\frac{1}{2}} \right) | 0 \leq \ell \leq L, 0 \leq j \leq J-1 \right\}$$
$$\tau_h^{H} := \left\{ \left(x_{\ell+\frac{1}{2}}, y_{j+\frac{1}{2}} \right) | 0 \leq \ell \leq L-1, 0 \leq j \leq J-1 \right\}$$
$$\tau_t^{E} := \left\{ \left(t^n \right) | 0 \leq n \leq N \right\}$$
$$\tau_t^{H} := \left\{ \left(t^{n+\frac{1}{2}} \right) | 0 \leq n \leq N-1 \right\}.$$

Let U be one of the field variables: H, E_x, E_y, $\vec{\alpha}_x$, $\vec{\alpha}_y$, $\vec{\beta}_x$, $\vec{\beta}_y$. Let (x_i, y_j) be a node on any discrete spatial mesh, and γ be either n or $n + \frac{1}{2}$ with $\gamma \leq N$.

We define the *grid functions* or the numerical approximations

$$U_{i,j}^{\gamma} \approx U(x_i, y_j, t^{\gamma}).$$

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$$U^{\gamma}_{i,j} \approx U(x_i, y_j, t^{\gamma}).$$

We define the centered temporal difference operator and a discrete time averaging operation as

$$\delta_t U^{\gamma}_{i,j} := \frac{U^{\gamma+\frac{1}{2}}_{i,j} - U^{\gamma-\frac{1}{2}}_{i,j}}{\Delta t}, \qquad \overline{U}^{\gamma}_{i,j} := \frac{U^{\gamma+\frac{1}{2}}_{i,j} + U^{\gamma-\frac{1}{2}}_{i,j}}{2}, \tag{5}$$

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and the centered spatial difference operators in the x and y direction, respectively as

$$\delta_x U^{\gamma}_{i,j} := \frac{U^{\gamma}_{i+\frac{1}{2},j} - U^{\gamma}_{i-\frac{1}{2},j}}{\Delta x}, \qquad \delta_y U^{\gamma}_{i,j} := \frac{U^{\gamma}_{i,j+\frac{1}{2}} - U^{\gamma}_{i,j-\frac{1}{2}}}{\Delta y}. \tag{6}$$

The Yee Scheme applied to the Maxwell-PC Lorentz yields

$$\mu_0\delta_t H^n_{\ell+\frac{1}{2},j+\frac{1}{2}} = \left(\delta_y E^n_{x_{\ell+\frac{1}{2},j+\frac{1}{2}}} - \delta_x E^n_{y_{\ell+\frac{1}{2},j+\frac{1}{2}}}\right) \tag{7a}$$

$$\epsilon_0\epsilon_\infty\delta_t E^{n+\frac{1}{2}}_{x_{\ell+\frac{1}{2},j}} = \delta_y H^{n+\frac{1}{2}}_{\ell+\frac{1}{2},j} - \overline{\vec{\beta}}^{n+\frac{1}{2}}_{0,x_{\ell+\frac{1}{2},j}} \tag{7b}$$

$$\epsilon_0\epsilon_\infty\delta_t E^{n+\frac{1}{2}}_{y_{\ell,j+\frac{1}{2}}} = -\delta_x H^{n+\frac{1}{2}}_{\ell,j+\frac{1}{2}} - \overline{\vec{\beta}}^{n+\frac{1}{2}}_{0,y_{\ell,j+\frac{1}{2}}} \tag{7c}$$

$$\delta_t \vec{\alpha}^{n+\frac{1}{2}}_{x_{\ell+\frac{1}{2},j}} = \overline{\vec{\beta}}^{n+\frac{1}{2}}_{x_{\ell+\frac{1}{2},j}} \tag{7d}$$

$$\delta_t \vec{\alpha}^{n+\frac{1}{2}}_{y_{\ell,j+\frac{1}{2}}} = \overline{\vec{\beta}}^{n+\frac{1}{2}}_{y_{\ell,j+\frac{1}{2}}} \tag{7e}$$

$$\delta_t \vec{\beta}^{n+\frac{1}{2}}_{x_{\ell+\frac{1}{2},j}} = -A\overline{\vec{\alpha}}^{n+\frac{1}{2}}_{x_{\ell+\frac{1}{2},j}} - 2\nu\overline{\vec{\beta}}^{n+\frac{1}{2}}_{x_{\ell+\frac{1}{2},j}} + \hat{e}_1\epsilon_0\omega_p^2\overline{E}^{n+\frac{1}{2}}_{x_{\ell+\frac{1}{2},j}} \tag{7f}$$

$$\delta_t \vec{\beta}^{n+\frac{1}{2}}_{y_{\ell,j+\frac{1}{2}}} = -A\overline{\vec{\alpha}}^{n+\frac{1}{2}}_{y_{\ell,j+\frac{1}{2}}} - 2\nu\overline{\vec{\beta}}^{n+\frac{1}{2}}_{y_{\ell,j+\frac{1}{2}}} + \hat{e}_1\epsilon_0\omega_p^2\overline{E}^{n+\frac{1}{2}}_{y_{\ell,j+\frac{1}{2}}}. \tag{7g}$$

Staggered L^2 normed spaces

Next, we define the L^2 normed spaces

$$\mathbb{V}_E := \left\{ \mathbf{F} : \tau_h^{E_x} \times \tau_h^{E_y} \longrightarrow \mathbb{R}^2 \mid \mathbf{F} = (F_{x_{l+\frac{1}{2},j}}, F_{y_{l,j+\frac{1}{2}}})^T, \|\mathbf{F}\|_E < \infty \right\} \quad (8)$$

$$\mathbb{V}_H := \left\{ U : \tau_h^{H} \longrightarrow \mathbb{R} \mid U = (U_{l+\frac{1}{2},j+\frac{1}{2}}), \|U\|_H < \infty \right\} \quad (9)$$

with the following discrete norms and inner products

$$\|\mathbf{F}\|_E^2 = \Delta x \Delta y \sum_{\ell=0}^{L-1} \sum_{j=0}^{J-1} \left(|F_{x_{\ell+\frac{1}{2},j}}|^2 + |F_{y_{\ell,j+\frac{1}{2}}}|^2 \right), \forall\ \mathbf{F} \in \mathbb{V}_E \quad (10)$$

$$(\mathbf{F}, \mathbf{G})_E = \Delta x \Delta y \sum_{\ell=0}^{L-1} \sum_{j=0}^{J-1} \left(F_{x_{\ell+\frac{1}{2},j}} G_{x_{\ell+\frac{1}{2},j}} + F_{y_{\ell,j+\frac{1}{2}}} G_{y_{\ell,j+\frac{1}{2}}} \right), \forall\ \mathbf{F}, \mathbf{G} \in \mathbb{V}_E \quad (11)$$

$$\|U\|_H^2 = \Delta x \Delta y \sum_{\ell=0}^{L-1} \sum_{j=0}^{J-1} |U_{\ell+\frac{1}{2},j+\frac{1}{2}}|^2, \forall\ U \in \mathbb{V}_H \quad (12)$$

$$(U, V)_H = \Delta x \Delta y \sum_{\ell=0}^{L-1} \sum_{j=0}^{J-1} U_{\ell+\frac{1}{2},j+\frac{1}{2}} V_{\ell+\frac{1}{2},j+\frac{1}{2}}, \forall\ U, V \in \mathbb{V}_H. \quad (13)$$

We define a space and inner product for the random polarization in vector notation, since $\vec{\boldsymbol{\alpha}}$ and $\vec{\boldsymbol{\beta}}$ are now $2 \times p+1$ matrices:

$$\mathbb{V}_\alpha := \left\{ \vec{\boldsymbol{\alpha}} : \tau_h^{E_x} \times \tau_h^{E_y} \longrightarrow \mathbb{R}^2 \times \mathbb{R}^{p+1} \;\middle|\; \vec{\boldsymbol{\alpha}} = [\boldsymbol{\alpha}_0, \ldots, \boldsymbol{\alpha}_p], \boldsymbol{\alpha}_k \in \mathbb{V}_E, \|\vec{\boldsymbol{\alpha}}\|_\alpha < \infty \right\}$$

where the discrete L^2 grid norm and inner product are defined as

$$\|\vec{\boldsymbol{\alpha}}\|_\alpha^2 = \sum_{k=0}^{p} \|\boldsymbol{\alpha}_k\|_E^2, \quad \forall\, \vec{\boldsymbol{\alpha}} \in \mathbb{V}_\alpha$$

$$(\vec{\boldsymbol{\alpha}}, \vec{\boldsymbol{\beta}})_\alpha = \sum_{k=0}^{p} \left(\boldsymbol{\alpha}_k, \boldsymbol{\beta}_k \right)_E, \quad \forall\, \vec{\boldsymbol{\alpha}}, \vec{\boldsymbol{\beta}} \in \mathbb{V}_\alpha.$$

We choose both spatial steps to be uniform and equal ($\Delta x = \Delta y = h$), and require that the usual CFL condition for two dimensions holds:

$$\sqrt{2} c_\infty \Delta t \leq h. \tag{14}$$

Theorem (Energy Decay for Maxwell-PC Lorentz-FDTD)

If the stability condition (14) is satisfied, then the Yee scheme for the 2D TE mode Maxwell-PC Lorentz system satisfies the discrete identity

$$\delta_t \mathcal{E}_h^{n+\frac{1}{2}} = \frac{-1}{\overline{\mathcal{E}}_h^{n+\frac{1}{2}}} \left\| \sqrt{\frac{2\nu}{\epsilon_0 \omega_p^2}} \overline{\vec{\beta}}^{n+\frac{1}{2}} \right\|_\alpha^2 \tag{15}$$

for all n where

$$\mathcal{E}_h^n = \left(\mu_0 (H^{n+\frac{1}{2}}, H^{n-\frac{1}{2}})_H + \|\sqrt{\epsilon_0\epsilon_\infty}\ \mathbf{E}^n\|_E^2 + \left\| \frac{A^{1/2}}{\sqrt{\epsilon_0\omega_p^2}} \vec{\alpha}^n \right\|_\alpha^2 + \left\| \sqrt{\frac{1}{\epsilon_0\omega_p^2}} \vec{\beta}^n \right\|_\alpha^2 \right)^{1/2} \tag{16}$$

defines a discrete energy.

In the above, A positive definite iff $r < m$, and assumed to have been symmetrized.
Note that $\|\vec{\alpha}\|_\alpha^2 \approx \|\mathbb{E}[\mathcal{P}]\|_2^2 + \|\mathrm{StdDev}(\mathcal{P})\|_2^2 = \mathbb{E}[\|\mathcal{P}\|_2^2] = \|\mathcal{P}\|_F^2$ so that this is a natural extension of the Maxwell-Random Lorentz energy.

Assuming a solution to the Maxwell-Random Lorentz system of the form $\mathbf{E} = \mathbf{E_0} exp(i(\omega t - \mathbf{k} \cdot \mathbf{x}))$, the following relation must hold.

Theorem

The dispersion relation for the Maxwell-Random Lorentz system is given by

$$\frac{\omega^2}{c^2}\epsilon(\omega) = k^2$$

where the expected complex permittivity is given by

$$\epsilon(\omega) = \epsilon_\infty + \omega_p^2 \mathbb{E}\left[\frac{1}{\omega_0^2 - \omega^2 - i2\nu\omega}\right].$$

Where $\mathbf{k} = [k_x, k_y, k_x]^T$ is the wave vector, $k^2 = k_x^2 + k_y^2 + k_z^2$ is the wavenumber (squared) and $c = 1/\sqrt{\mu_0 \epsilon_0}$ is the speed of light.

Theorem

The discrete dispersion relation for the Maxwell-PC FDTD Lorentz scheme is given by

$$\frac{\omega_\Delta^2}{c^2}\epsilon_\Delta(\omega) = k_\Delta^2$$

where the discrete expected complex permittivity is given by

$$\epsilon_\Delta(\omega) := \epsilon_\infty + \omega_{p,\Delta}^2 \hat{e}_1^T \left(A_\Delta - \omega_\Delta^2 I - i2\nu_\Delta\omega_\Delta I\right)^{-1} \hat{e}_1$$

and the discrete wavenumber is given by

$$k_\Delta := \sqrt{x_\Delta^2 + y_\Delta^2},$$

with

$$x_\Delta := \frac{2}{\Delta x}\sin\left(\frac{k_{x,\Delta}\Delta x}{2}\right), \quad x_\Delta := \frac{2}{\Delta y}\sin\left(\frac{k_{y,\Delta}\Delta y}{2}\right)\ldots$$

Theorem (Continued)

and the discrete PC matrix and discrete damping are given by

$$A_\Delta := \cos^2(\omega\Delta t/2)A, \quad \nu_\Delta := cos\left(\frac{\omega\Delta t}{2}\right)\nu.$$

Similarly,

$$\omega_\Delta := \frac{2}{\Delta t}\sin\left(\frac{\omega\Delta t}{2}\right), \quad \omega_{p,\Delta} := cos\left(\frac{\omega\Delta t}{2}\right)\omega_p.$$

Dispersion Error

The exact dispersion relation can be compared with a discrete dispersion relation to determine the amount of dispersion error.
We define the phase error Φ for a scheme applied to a model to be

$$\Phi = \left| \frac{k_{EX} - k_{\Delta}}{k_{EX}} \right|, \tag{17}$$

where the numerical wave number k_{Δ} is implicitly determined by the corresponding discrete dispersion relation and k_{EX} is the exact wave number for the given model.

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where the numerical wave number k_{Δ} is implicitly determined by the corresponding discrete dispersion relation and k_{EX} is the exact wave number for the given model.

- We wish to examine the phase error as a function of ω in the range around $\overline{\omega}_0$. Δt is determined by $h := \overline{\omega}_0 \Delta t/(2\pi)$, while $\Delta x = \Delta y$ are determined by the CFL condition.
- We assume a uniform distribution and the following parameters Lorentz material:

$$\epsilon_\infty = 1, \quad \epsilon_s = 2.25, \quad \nu = 2.8 \times 10^{15} \text{ 1/sec}, \quad \overline{\omega}_0 = 4 \times 10^{16} \text{ rad/sec}.$$

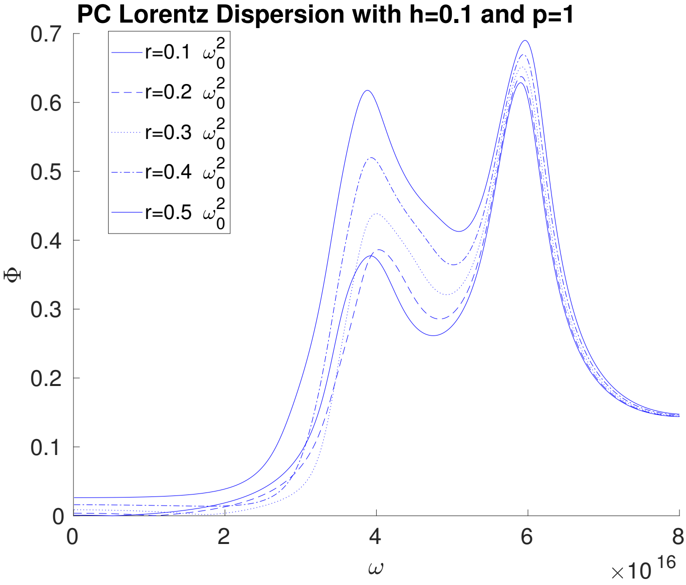

Figure: Plots of phase error at $\theta = 0$.

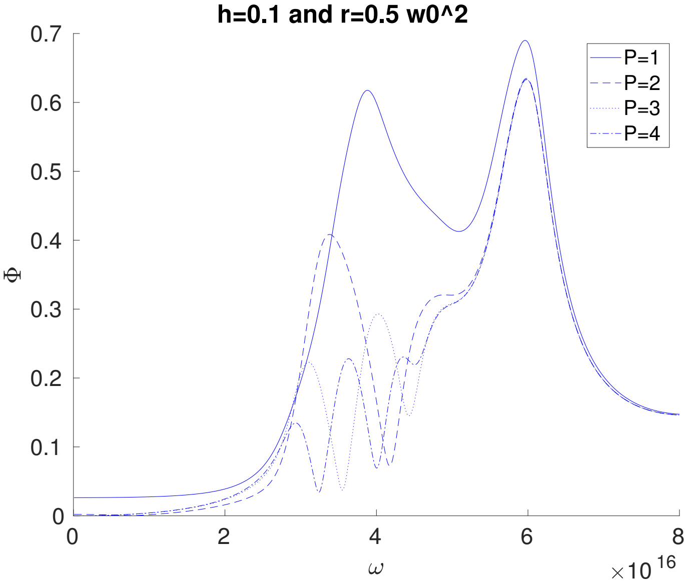

Figure: Plots of phase error at $\theta = 0$.

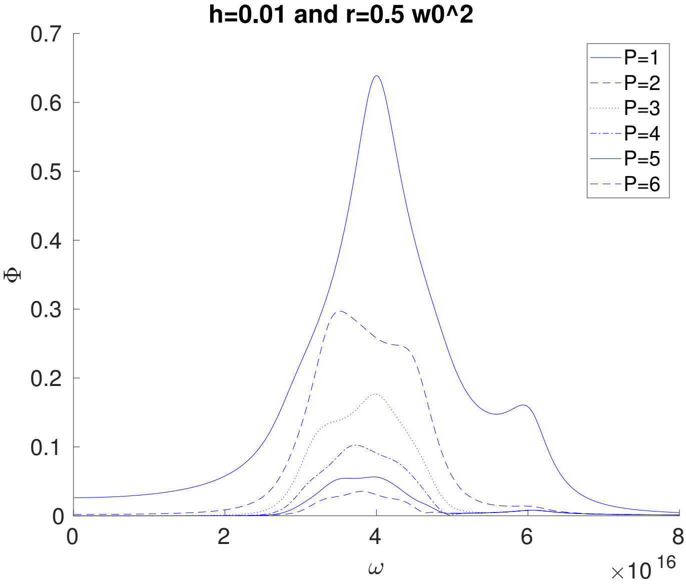

Figure: Plots of phase error at $\theta = 0$.

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