Optimization under uncertainty in magnetohydrodynamic generators

Nathan L. Gibson

Professor
Department of Mathematics

Oregon State University

Scientific Computing and Numerical Analysis
2021 SIAM PNW Conference
May 21, 2022

Acknowledgments

Collaborators

- Dr. Evan Rajbhandari, OSU
- Dr. Rigel Woodside, NETL

Funding

- NETL & ORISE [1]
- NSF grant DMS-2012882

[1] The work was supported by the U.S. Department of Energy's Offshore Research Program. This project was also supported by an appointment to the Science Education Programs at the National Energy Technology Laboratory (NETL), administered by ORAU through the U.S. Department of Energy Oak Ridge Institute for Science and Education.

Outline

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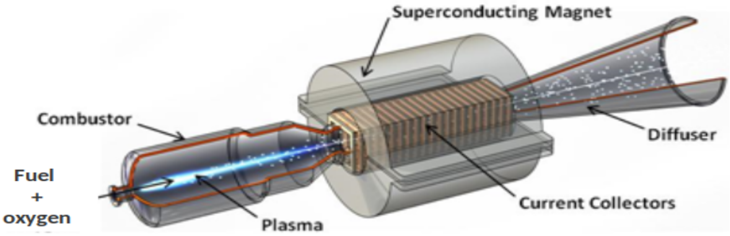

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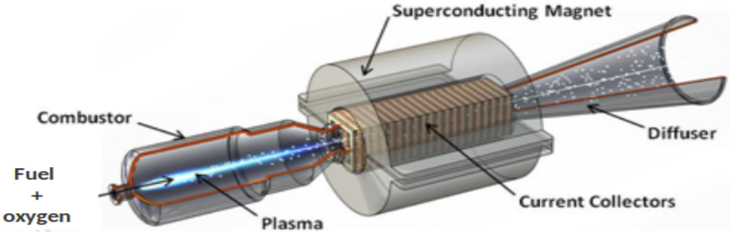

[2]Image courtesy R. Woodside

Outline

Governing Equations: Maxwell's Equations

Maxwell's Equations

$$\frac{\partial \mathbf{D}}{\partial t} - \nabla \times \mathbf{H} = \mathbf{J} \quad (1a)$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0 \quad (1b)$$

$$\nabla \cdot \mathbf{B} = \nabla \cdot \mathbf{E} = 0 \quad (1c)$$

$$\mathbf{E}(0, \mathbf{x}) = 0 \quad (1d)$$

B: Magnetic field
D: Displacement field
H: Magnetizing field
E: Electric field

Electrical Relations

Constitutive Laws

$$\mathbf{D}(t,\mathbf{x}) = \epsilon_0\epsilon_r(\mathbf{x})\mathbf{E}(t,\mathbf{x}) + \mathbf{P}(t,\mathbf{x}) \tag{2a}$$

$$\mathbf{B}(t,\mathbf{x}) = \mu_0\mathbf{H}(t,\mathbf{x}) + \mathbf{M}(t,\mathbf{x}) \tag{2b}$$

Generalized Ohm's Law

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{u}\times\mathbf{B}) + \frac{\beta_e}{||\mathbf{B}||}(\mathbf{J}\times\mathbf{B}) + \frac{\beta_i}{||\mathbf{B}||^2}((\mathbf{J}\times\mathbf{B})\times\mathbf{B}), \tag{3}$$

μ_0: Permeability of free-space
$\epsilon_0\epsilon_r$: Permittivity
P: Polarization
M: Magnetization
σ: Conductivity
u : Fluid velocity field

Hall Parameter

$$\beta_e = \mu_e||\mathbf{B}||$$

Ion-slip parameter

$$\beta_i = \mu_e\mu_i||\mathbf{B}||^2$$

Governing Equations: Maxwell's for MHD

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Assume
- all functions are steady state,
- the magnetic field, **B**, is given,
- the fluid-flow, **u**, are prescribed.

Then the governing equations for an MHD channel are given by

$$\nabla \times \mathbf{E} = 0 \qquad \text{on } \Omega, \tag{4a}$$

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}) + \frac{\beta_e}{||\mathbf{B}||}(\mathbf{J} \times \mathbf{B}) + \frac{\beta_i}{||\mathbf{B}||^2}\big((\mathbf{J} \times \mathbf{B}) \times \mathbf{B}\big) \qquad \text{on } \Omega, \tag{4b}$$

$$\nabla \cdot \mathbf{J} = 0 \qquad \text{on } \Omega. \tag{4c}$$

Explicit Ohm's Law

Using a matrix representation of the cross-product and algebra, (3) can be rewritten as

$$\mathbf{J} = \sigma(\mathcal{I} - \beta_e[\mathbf{B}]_\times - \beta_i[\mathbf{B}]_\times^2)^{-1}(\mathbf{E} + \mathbf{u} \times \mathbf{B}) = \underline{\overline{\sigma}}(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

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We define $\mathbf{J}_i$ as

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Employing the divergence-free condition (4c), implies

$$-\nabla \cdot \mathbf{J}_i = \nabla \cdot \underline{\overline{\sigma}}(\mathbf{u} \times \mathbf{B}), \tag{6}$$

Mixed-Poisson's Equations

Combining (5) and (6), and writing $\mathbf{E} = \nabla \mathcal{V}$, we have

$$\underline{\overline{\sigma}}^{-1} \mathbf{J}_i - \nabla \mathcal{V} = 0, \tag{7a}$$

$$-\nabla \cdot \mathbf{J}_i = \nabla \cdot \left(\underline{\overline{\sigma}} (\mathbf{u} \times \mathbf{B}) \right). \tag{7b}$$

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Thus, solving (4) is equivalent to a system of mixed Poisson equations, and employing that

$$\mathbf{J} = \mathbf{J}_i + \underline{\overline{\sigma}}(\mathbf{u} \times \mathbf{B}) \tag{8a}$$

$$\mathbf{E} = \nabla \mathcal{V}. \tag{8b}$$

(We write the system in this way to be able to apply the Babuska-Brezzi-Kovalevskaya Theorem for existence and uniqueness of solutions.)

Strong Form to Weak Form

We define the following spaces for our solutions to (7).

$$\mathbf{J}_i \in V(\Omega) = \left\{ f \in L^2(\Omega) : f \cdot \mathbf{n} = 0 \text{ on } \Gamma \right\},$$

and

$$\mathcal{V} \in W(\Omega) := W_0^{1,2}(\Omega) = \left\{ f \in H^1(\Omega) : T(f) = 0 \right\},$$

where $T(f)$ is the trace of f on Γ.

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Then (7) is weakly-equivalent to

$$\int_\Omega (\underline{\overline{\sigma}}^{-1}\mathbf{J}) \cdot \phi - \int_\Omega \nabla \mathcal{V} \cdot \phi = 0 \;\forall \phi \in V(\Omega), \tag{9a}$$

$$-\int_\Omega \mathbf{J}_i \cdot \nabla\psi = \int_\Omega \left(\underline{\overline{\sigma}}\mathbf{u} \times \mathbf{B} \right) \cdot \nabla\psi \;\forall \psi \in W(\Omega) \tag{9b}$$

Well-posedness of this forward problem is established within [3].

Deterministic Parameter Identification

We wish to formulate the parameter estimation problem for this system, with potential unknowns

- Fluid flow $\mathbf{u}$
- Plasma conductivity σ
- Electron mobility μ_e
- Ion mobility μ_i

Well-posedness of this inverse problem is also established within [3].

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Well-posedness of this inverse problem is also established within [3].

However, we further allow for these parameters to be random.

Outline

Uncertain Kinematic MHD Equations

Find $\mathbf{J}_i \in \overline{V}, \mathcal{V} \in \overline{W}$ that satisfy

$$\mathbb{E}\left[\int_D \underline{\sigma}^{-1}\mathbf{J}_i \cdot \phi\right] - \mathbb{E}\left[\int_D \nabla\mathcal{V} \cdot \phi\right] = 0,\ \forall \phi \in \overline{V}, \tag{10a}$$

$$-\mathbb{E}\left[\int_D \mathbf{J}_i \cdot \nabla\psi\right] = \mathbb{E}\left[\int_D \left(\underline{\sigma}(\mathbf{u} \times \mathbf{B})\right) \cdot \nabla\psi\right],\ \forall \psi \in \overline{W} \tag{10b}$$

where

$$\overline{V} = V \times L^2(\Omega),\ \overline{W} = W \times L^2(\Omega).$$

Uncertain Kinematic MHD Operator Form

Find $\mathbf{J}_i \in \overline{V}, \mathcal{V} \in \overline{W}$ such that

$$\overline{\mathcal{A}}(\mathbf{J}_i) + \overline{\mathcal{B}}'(\mathcal{V}) = 0 \ \in \overline{V}', \tag{11a}$$

$$\overline{\mathcal{B}}(\mathbf{J}_i) = \overline{G} \ \in \overline{W}' \tag{11b}$$

Application of the Babuska-Brezzi-Kovalevskaya Theorem results in the well-posedness of the system, following a similar approach to that of the deterministic version [2].
Well-posedness of this random inverse problem is also established.

Outline

Notation

$$q = (\mathbf{u}, \sigma, \mu, \mu_i).$$

$$U(\mathbf{x}; q) = \begin{bmatrix} \mathbf{J}_i(\mathbf{x}; q) \\ \mathcal{V}(\mathbf{x}; q) \end{bmatrix}, \; A(q) = \begin{bmatrix} \overline{\mathcal{A}}(q) & \overline{\mathcal{B}}'(q) \\ \overline{\mathcal{B}}(q) & 0 \end{bmatrix}, \; F(q) = \begin{bmatrix} 0 \\ G(q) \end{bmatrix}$$

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System (11), can be written as

$$AU = F. \tag{12}$$

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Let $\overline{H} = \overline{V} \times \overline{W}$, with norm $||\cdot||^2_{\overline{H}} = ||\cdot||^2_{\overline{V}} + ||\cdot||^2_{\overline{W}}$.

Deterministic Identification Problem

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(ID) $$\min_{q \in Q} J(q) := ||U(\cdot; q) - D||_H^2$$

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(UID)

$$\min_{P_q \in \mathcal{P}(Q)} J(P_q) := \mathbb{E}\left[||U(q) - \mathcal{U}||_H^2 \middle| P_q \right].$$

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- Distribution space, $\mathcal{P}(Q)$, with M poles

DRUID

Let

$$\widetilde{Q} \subset Q, \text{ compact and } \widetilde{Q}^N = \widetilde{Q} \cap \mathbb{P}^N$$

Then for $P \in \mathcal{P}_M(\widetilde{Q}_N)$, $M \geq N$, the expected value of a random process $f : \widetilde{Q}_N \to H$ is given by

$$\mathbb{E}[f|P] = \int_{\widetilde{Q}} f(q)\, dP(q) = \sum_{j=1}^{M} f(q_j^N) p_j$$

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For fixed $K, M, N \in \mathbb{N}$, and data $\{\mathcal{U}_k\}_{k=1}^K$, our dimension reduced uncertain identification problem becomes

$$\text{(DRUID)} \qquad \min_{P \in \mathcal{P}_M(\widetilde{Q}_N)} J_{M,N}^K(P) = \sum_{j=1}^{M} \sum_{k=1}^{K} \left| U(x_k, q_j^N) - \mathcal{U}_k \right|^2 p_j$$

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$$\mathcal{V}(\mathbf{x},\omega) = \mathcal{V}(\mathbf{x},\psi_1(\omega),\ldots,\psi_{\mathcal{M}}(\omega))$$

Let $\Gamma_k := \psi_k(\Omega)$, and $\Gamma := \prod_{k=1}^{\mathcal{M}} \Gamma_k$.

Notation

- $m = [m_1, \ldots, m_{\mathcal{M}}]$ an array of indices
- M_k, the number of sample points in the direction $\Gamma_k,\ k = 1, \ldots, \mathcal{M}$
- $M = \prod_{k=1}^{\mathcal{M}} M_k$
- $\{r_k^j\}_{j=0}^{M_k}$, a basis of orthogonal Chebyshev polynomials that satisfy

$$\int_{\Gamma_k} r_k^j r_k^l P_Y(y)\ dy = \delta_{jl} w_k^j, \text{ where } w_k^j := \int_{\Gamma_k} \left(r_k^j(y_m)\right)^2 P_Y(y_m)\ dy$$

- $y_k^{m_k}$ for $m_k = 1, \ldots, M_k,\ k = 1, \ldots, \mathcal{M}$, the m^{th} unique zero in the direction Γ_k
- $y_m = [y_1^{m_1}, \ldots, y_M^{m_{\mathcal{M}}}]$, a collection of zeroes in each random direction
- $r_m(y) = \prod_{j=1}^{\mathcal{M}} r_j^{m_j}(y)$, the product of the polynomials in each direction

Polynomial chaos expansion of $\mathcal{V}$:

$$\mathcal{V}^{h,M}(\mathbf{x}, y) = \sum_{m_1=1}^{M_1} \dots \sum_{m_{\mathcal{M}}=1}^{M_{\mathcal{M}}} \mathcal{V}^h(\mathbf{x}, y_m) r_m(y). \tag{13}$$

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For notational convenience, we write

$$\mathbb{E}\left[\mathcal{V}^h | P_Y^M\right] = \mathbb{E}\left[\mathcal{V}^{h,M}\right]$$

- *Expected* solution, $\mathbb{E}\left[\mathcal{V}^{h,M}\right]$

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Parameter	Mean	Standard Deviation
μ_e	10/6	1/6
$\mathbf{u}_x$	1600	160

Table 1: Parameter distributions used in the determining *expected* solutions

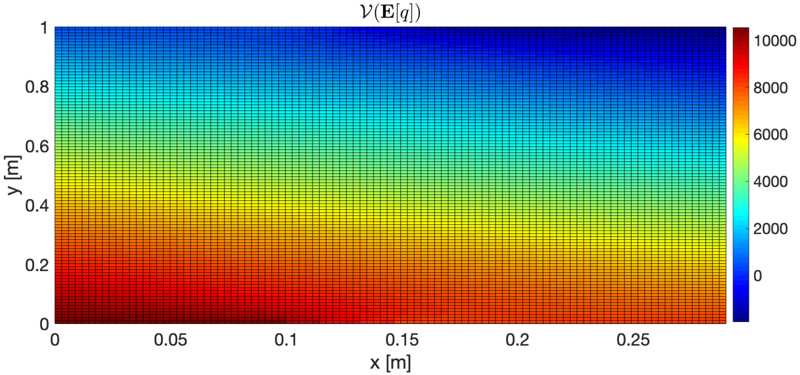

Figure 1: Deterministic $\mathcal{V}$

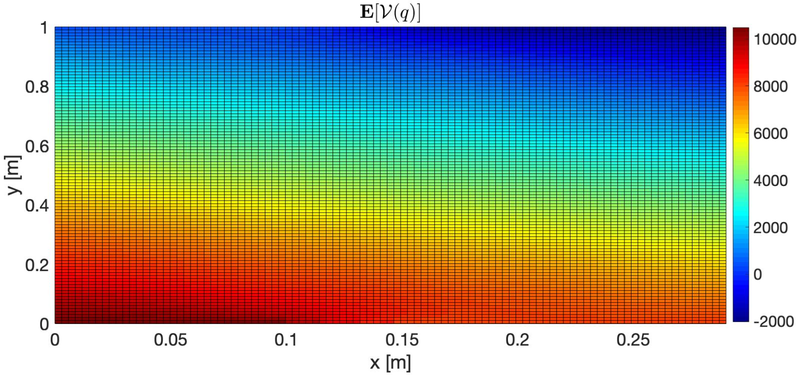

Figure 2: Expected $\mathcal{V}$

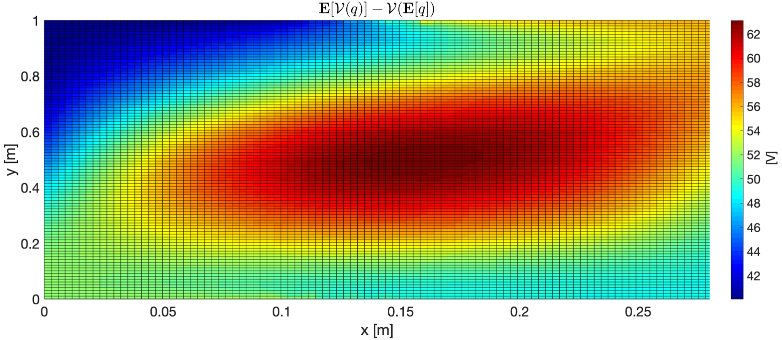

Figure 3: Deterministic $\mathcal{V}$ - Expected $\mathcal{V}$

Comparing spatial functions $\mathbb{E}[\mathcal{V}]$

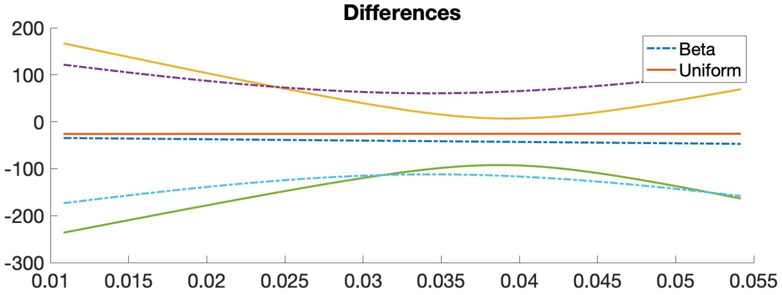

Figure 4: Comparing the distributions of $\mathcal{V}$ under an assumption of beta and uniform parameter distribution. Associated lines are $\mathbb{E}[U(q)] - U(\mathbb{E}[q])$, with upper and lower lines representing one standard deviation away.

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Covariance

Suppose Y is described by

$$\mathcal{C} : D \times D \to \mathbb{R}, \quad \mathcal{C}(\mathbf{t}, \mathbf{s}) = \mathrm{Cov}\big(Y(\mathbf{t}, \omega), Y(\mathbf{s}, \omega)\big)$$

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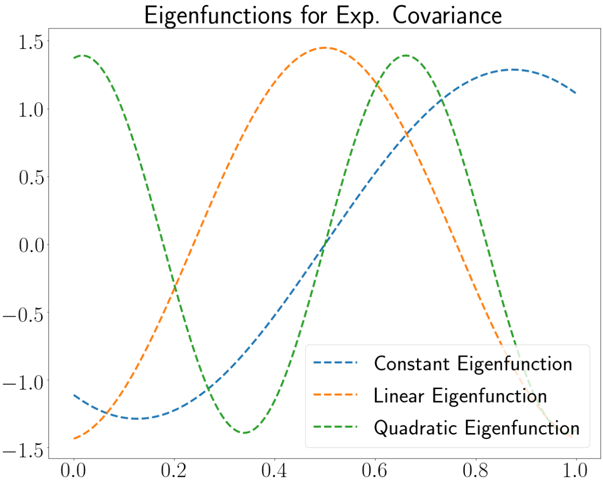

Figure 5: Eigenfunctions for exponential covariance, i.e. ϕ_j, for $j = 0, 1, 2$.

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Assume $\mathcal{C}$ has known eigenfunctions, $\{\phi_j\}_{j=1}^{\infty}$, which satisfy

$$\lambda_j \phi_j(s) = \int_D \mathcal{C}(\mathbf{x}, s)\phi_j(\mathbf{x})\; d\mathbf{x}. \tag{15}$$

Define $\big\{\psi_j : \Omega \to \mathbb{R},\ \mathbb{E}[\psi_j] = 0,\ \mathrm{var}[\psi_j] = 1\big\}_{j=0}^{\infty}$, *uncorrelated* w.r.t. $\mathcal{C}$,

$$C(\psi_j, \psi_k) = \delta_{j,k}.$$

Karhunen-Loève expansion (KLE)

KLE:

$$Y(\mathbf{x},\omega) = \mathbb{E}[Y](\mathbf{x}) + \sum_{j=0}^{\infty} \sqrt{\lambda_j}\phi_j(\mathbf{x})\psi_j(\omega). \tag{16}$$

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for some set $\{a_k : \text{ for } k = 0, \ldots, N, \ a_k \in \mathbb{R}\}$.

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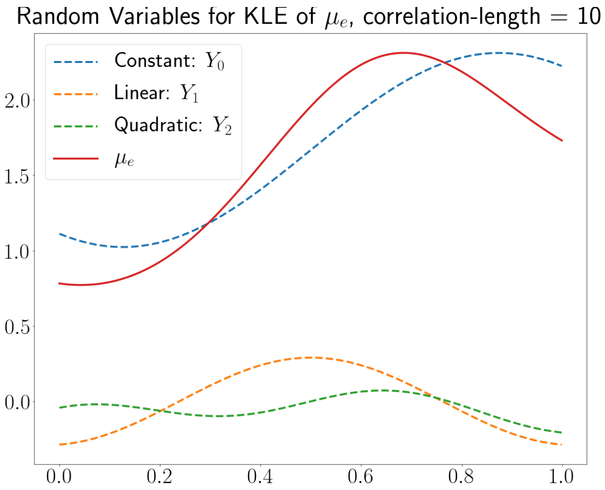

Figure 6: (Dashed) Individual random processes and (solid) quadratic μ_e.

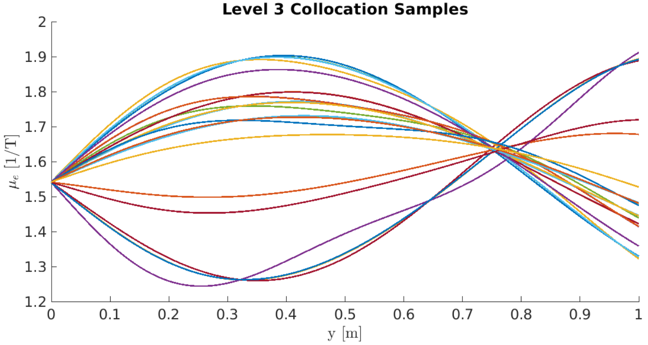

Figure 7: Collocation sample points of the μ_e under the KLE, with an Fejér grid of level$= 3$.

Outline

Distributions

Table 2: Designated 'true' deterministic coeffecients and eigenvalues, for use in the numerical implementation of (DRUID)

Parameter	True Value
a_0	10/6
λ_0	0.5
a_1	0
λ_1	0.02
a_2	-1.5
λ_2	0.016

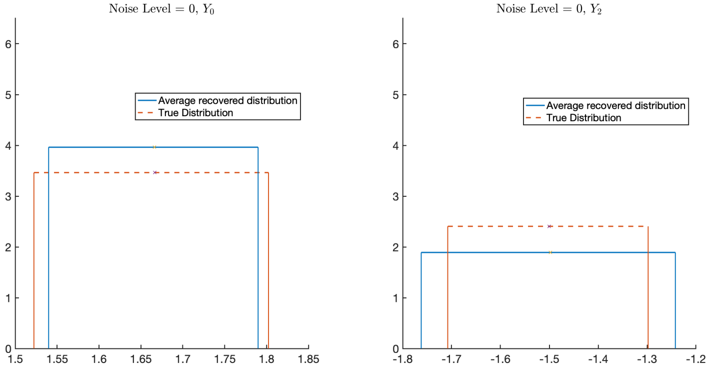

Figure 8: Demonstrations of Distributional recovery, at $\mathbf{x} = (0.05, 0.5, 0.05)$, for **quadratic** μ_e, with an assumed **uniform** distribution covariance structure, and with the **partial** domain available for data, at a noise level of 0.

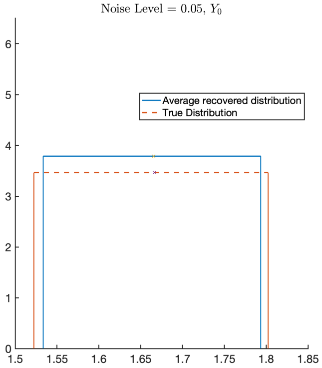

Figure 9: Demonstrations of Distributional recovery, at $\mathbf{x} = (0.05, 0.5, 0.05)$, for **quadratic** μ_e, with an assumed **uniform** distribution covariance structure, and with the **partial** domain available for data, at a noise level of 0.05.

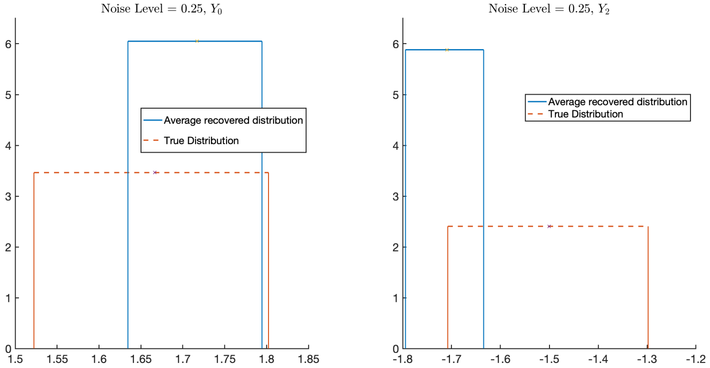

Figure 10: Demonstrations of Distributional recovery, at $\mathbf{x} = (0.05, 0.5, 0.05)$, for **quadratic** μ_e, with an assumed **uniform** distribution covariance structure, and with the **partial** domain available for data, at a noise level of 0.25.

Table 3: Shape parameters for the random variables with an assumed beta distribution function.

Y_0	Y_1	Y_2
$(2, 2)$	$(3, 1)$	$(4, 2)$

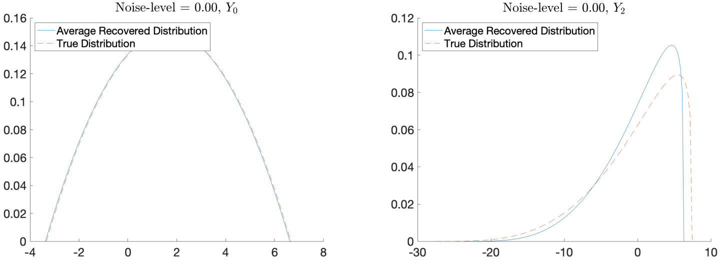

Figure 11: Demonstrations of Distributional recovery, at $\mathbf{x} = (0.05, 0.5, 0.05)$, for **quadratic** μ_e, with an assumed **uniform** distribution covariance structure, and with the **partial** domain available for data, at a noise level of 0.

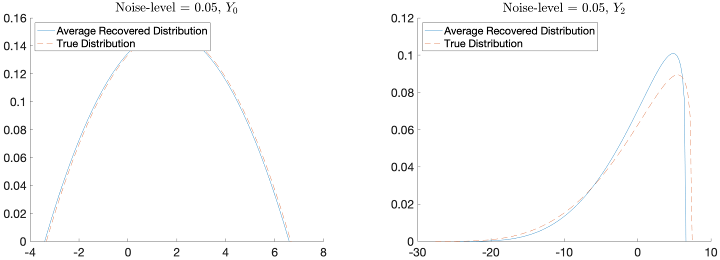

Figure 12: Demonstrations of Distributional recovery, at $\mathbf{x} = (0.05, 0.5, 0.05)$, for **quadratic** μ_e, with an assumed **uniform** distribution covariance structure, and with the **partial** domain available for data, at a noise level of 0.05.

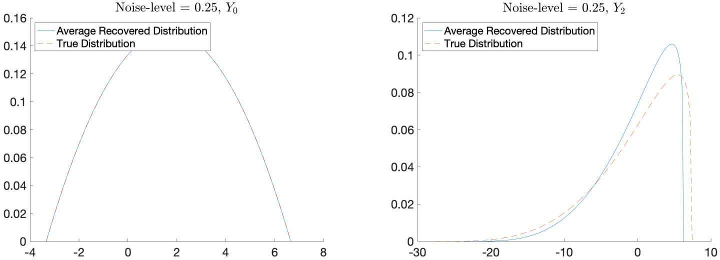

Figure 13: Demonstrations of Distributional recovery, at $\mathbf{x} = (0.05, 0.5, 0.05)$, for **quadratic** μ_e, with an assumed **uniform** distribution covariance structure, and with the **partial** domain available for data, at a noise level of 0.25.

Outline

Summary

- Introduction of uncertainty
- DRUID
- Stochastic Collocation
- Karhunen-Loève Expansion
- Numerical method for UPE
- Numerical results for uniform and beta distributions.

Bibliography I

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[4] E. Rajbhandari, N.L. Gibson, and C.R. Woodside. "Quantifying uncertainty with stochastic collocation in the kinematic magentohydrodynamic framework". In: *Journal of Computational Physics: Conference Series* (2022).