7. Find the general solution of
\[ y'' + 4y' + 4y = t^{-2}e^{-2t}, \quad t > 0. \]

**Sol.** First solve for the homogeneous solutions by writing the characteristic equation:
\[ r^2 + 4r + 4 = 0 \]
which can be factored into \((r + 2)^2 = 0\).

Thus \(r = -2\) is a repeated root. Therefore the solutions to the homogeneous problem are
\[ y_1 = e^{-2t}, \quad y_2 = te^{-2t}. \]

To solve the non-homogeneous problem, we assume the form \(Y = u_1(t)y_1 + u_2(t)y_2\), where \(u_1\) and \(u_2\) solve the following first order ODEs:
\[ u_1' = -\frac{y_2g(t)}{W(y_1, y_2)} \quad u_2' = \frac{y_1g(t)}{W(y_1, y_2)} \]
where \(g(t) = t^{-2}e^{-2t}\) is the forcing term in the original second order ODE. The Wronskian is given by
\[ W(y_1, y_2)(t) = y_1y_2' - y_2y_1' = (e^{-2t})(-2te^{-2t} + e^{-2t}) - (te^{-2t})(-2e^{-2t}) = e^{-4t}. \]

The ODEs become
\[ u_1' = -\frac{(te^{-2t})(t^{-2}e^{-2t})}{e^{-4t}} = -\frac{1}{t} \quad u_2' = \frac{(e^{-2t})(t^{-2}e^{-2t})}{e^{-4t}} = \frac{1}{t^2}. \]

Integrating we find
\[ u_1 = -\ln(t) \quad u_2 = \frac{1}{t}. \]

Thus our particular solution is
\[ Y = u_1(t)y_1 + u_2(t)y_2 \]
\[ = (-\ln(t))(e^{-2t}) + \left(-\frac{1}{t}\right)(te^{-2t}) \]
\[ = -\ln(t)e^{-2t} - e^{-2t}. \]

Therefore the general solution is
\[ y = c_1e^{-2t} + c_2te^{-2t} - \ln(t)e^{-2t} - e^{-2t}. \]
Note that the last term can be absorbed into the first if we let $\tilde{c}_1 = c_1 - 1$, so that finally the general solution can be written

$$y = \tilde{c}_1 e^{-2t} + c_2 t e^{-2t} - \ln(t) e^{-2t}.$$  

In this case we didn’t need the second term in the particular solution, i.e., we could have written it as $Y = -\ln(t) e^{-2t}$. But as any particular solution will do, we still get the same general solution.

Ex 2. Find the general solution of

$$t^2 y'' + ty' - 4y = \ln(t)$$

given that the homogeneous solutions are $y_1 = t^2$ and $y_2 = t^{-2}$ (Note: we showed in class how to solve for these homogeneous solutions based on the methods outlined in Problems 38-42 of Section 3.4.)

Sol. First we must rewrite the ODE in standard form by dividing both sides by $t^2$:

$$y'' + y'/t - 4y/t^2 = \ln(t)/t^2$$

To solve the non-homogeneous problem, we assume the form $Y = u_1(t)y_1 + u_2(t)y_2$, where $u_1$ and $u_2$ solve the following first order ODEs:

$$u_1' = -\frac{y_2 g(t)}{W(y_1, y_2)}$$
$$u_2' = \frac{y_1 g(t)}{W(y_1, y_2)}$$

where $g(t) = \ln(t)/t^2$ is the forcing term in the ODE. The Wronskian is given by

$$W(y_1, y_2)(t) = y_1 y_2' - y_2 y_1' = (t^2)(-2t^{-3}) - (t^{-2})(2t) = -4/t.$$  

The ODEs become

$$u_1' = -\frac{(t^{-2})(\ln(t))}{t^2(-4/t)} = \ln(t) \frac{4t^3}{4}$$
$$u_2' = \frac{(t^2)(\ln(t))}{t^2(-4/t)} = -\frac{t \ln(t)}{4}.$$  

Integrating we find

$$u_1 = -\frac{2 \ln(t) + 1}{16t^2}$$
$$u_2 = \frac{1 - 2 \ln(t)}{16} t^2.$$  

Thus our particular solution is

$$Y = u_1(t)y_1 + u_2(t)y_2 = \left(-\frac{2 \ln(t) + 1}{16t^2}\right) (t^2) + \left(\frac{1 - 2 \ln(t)}{16}\right) (t^{-2}) = -\ln(t)/4.$$  

Therefore the general solution is

$$y = c_1 t^2 + c_2 t^{-2} - \ln(t)/4.$$