8. Determine whether the following functions are linearly independent. If not, find a linear relation among them.

\[ f_1(t) = 2t - 3, \quad f_2(t) = 2t^2 + 1, \quad f_3(t) = 3t^2 + t. \]

**Sol.** The Wronskian is

\[
W(f_1, f_2, f_3) = \begin{vmatrix}
2t - 3 & 2t^2 + 1 & 3t^2 + t \\
2 & 4t & 6t + 1 \\
0 & 4 & 6
\end{vmatrix}
= -4 \begin{vmatrix}
2t - 3 & 3t^2 + t \\
2 & 6t + 1
\end{vmatrix} + 6 \begin{vmatrix}
2t - 3 & 2t^2 + 1 \\
2 & 4t
\end{vmatrix}
= -4 [(12t^2 - 16t - 3) - (6t^2 + 2t)] + 6 [(8t^2 - 12t) - (4t^2 + 2)]
= -24t^2 + 72t + 12 + 24t^2 - 72t - 12
= 0
\]

Thus the functions are linearly dependent. To find the relationship, solve for \(a\) and \(b\) in the following

\[ f_1 = af_2 + bf_3 \]
\[ 2t - 3 = 2at^2 + a + 3bt^2 + bt. \]

Thus \(a = -3\) and \(b = 2\). If the functions had been independent, we could not have found \(a\) and \(b\) to satisfy the above equation.

15. Verify that the given solutions are linearly independent by determining the Wronskian.

\[ xy''' - y'' = 0, \quad t > 0; \quad 1, \quad x, \quad x^3. \]

**Sol.** The Wronskian is

\[
W(y_1, y_2, y_3) = \begin{vmatrix}
1 & x & x^3 \\
0 & 1 & 3x^2 \\
0 & 0 & 6x
\end{vmatrix} = 6x \begin{vmatrix}
1 & x \\
0 & 1
\end{vmatrix} = 6x.
\]

Thus \(W(y_1, y_2, y_3) \neq 0\) for \(x > 0\). The solutions form a fundamental set.