1. (#16.) Find the general solution of

\[ y^{(4)} - 5y'' + 4y = 0. \]

\textbf{Sol.} First write the characteristic equation:

\[ Z(r) = r^4 - 5r^2 + 4 = 0 \]

which can be factored into

\[ (r^2 - 1)(r^2 - 4) = 0, \]

which in turn can be factored into

\[ (r - 1)(r + 1)(r - 2)(r + 2) = 0. \]

The roots are \( \pm 1 \) and \( \pm 2 \), thus the general solution to this homogeneous problem is

\[ y = c_1 e^t + c_2 e^{-t} + c_3 e^{2t} + c_4 e^{-2t} \]
2. Find the general solution of 
\[ y^{(4)} + 5y'' + 4y = 0. \]

**Sol.** First write the characteristic equation:
\[ Z(r) = r^4 + 5r^2 + 4 = 0 \]
which can be factored into 
\[ (r^2 + 1)(r^2 + 4) = 0. \]
The roots are ±i and ±2i, thus the general solution to this homogeneous problem is
\[ y = c_1 \cos(t) + c_2 \sin(t) + c_3 \cos(2t) + c_4 \sin(2t). \]

3. Consider the previous problem with initial conditions given by 
\[ y(0) = 1, \quad y'(0) = 1, \quad y''(0) = -1, \quad y'''(0) = -1, \]
find the solution to the IVP. **Sol.** The initial conditions impose the following constraints on the constants of integration
\[
\begin{align*}
  c_1 + c_3 & = 1 \quad (1a) \\
  c_2 + 2c_4 & = 1 \quad (1b) \\
  -c_1 - 4c_3 & = -1 \quad (1c) \\
  -c_2 + 8c_4 & = -1. \quad (1d)
\end{align*}
\]
Adding equations (1a) and (1c) we see that \( c_3 = 0 \). Adding (1b) and (1d) we see that \( c_4 = 0 \). This leaves \( c_1 = 1 \) and \( c_2 = 1 \). Therefore the solution to the IVP is
\[ y = \cos(t) + \sin(t). \]
4. Find the general solution of

\[ 9 y^{(4)} + 12 y''' + 82 y'' - 52 y' + 40y = 0 \]

**Sol.** First write the characteristic equation:

\[ Z(r) = 9r^4 + 12r^3 + 82r^2 - 52r + 40 \]

which can be factored into

\[ (9r^2 - 6r + 4)(r^2 + 2r + 10) = 0. \]

The roots are

\[ r_{1,2} = \frac{1 \pm \sqrt{3}i}{3} \quad r_{3,4} = -1 \pm 3i, \]

thus the general solution to this homogeneous problem is

\[ y = e^{t/3} \left( c_1 \cos\left( \frac{\sqrt{3}t}{3} \right) + c_2 \sin\left( \frac{\sqrt{3}t}{3} \right) \right) + e^{-t} \left( c_3 \cos(3t) + c_4 \sin(3t) \right). \]
5. Find the general solution of

$$y^{(4)} + 4y'' + 4y = 0.$$ 

**Sol.** First write the characteristic equation:

$$Z(r) = r^4 + 4r^2 + 4 = 0$$

which can be factored into

$$(r^2 + 2)^2 = 0.$$ 

The roots are $\pm \sqrt{2}i$ and each are double roots, thus the general solution to this homogeneous problem is

$$y = c_1 \cos(\sqrt{2}t) + c_2 \sin(\sqrt{2}t) + c_3 t \cos(\sqrt{2}t) + c_4 t \sin(\sqrt{2}t).$$