1. Find the solution of the initial value problem

\[ y'' + y = 3 \sin(2t), \quad y(0) = 2, \quad y'(0) = 1 \]

using the method of Laplace Transforms.

**Sol.** As a similar example is also solved in the book, I will point out where this approach differs from their solution.

- The first step is always to *transform both sides of the ODE* which results in

\[ s^2 Y(s) - sy(0) - y'(0) + Y(s) = 3 \frac{2}{s^2 + 2^2}, \]

where \( Y(s) = \mathcal{L}\{y\} \).

- Next we *substitute in the initial conditions* to get

\[ s^2 Y(s) - 2s - 1 + Y(s) = \frac{6}{s^2 + 4}. \]
Now we have an algebraic equation which can be solved for \( Y(s) \), as follows

\[
(s^2 + 1)Y(s) = 2s + 1 + \frac{6}{s^2 + 4}
\]

(Note that I have deliberately left the right hand side unsimplified, this differs from the book’s approach, and in my opinion, makes the upcoming partial fractions step easier)

\[
Y(s) = \frac{2s}{s^2 + 1} + \frac{1}{s^2 + 1} + \frac{6}{(s^2 + 1)(s^2 + 4)}. \tag{1}
\]

Only the last term above is not in a form which is easy to apply the inverse Laplace transform on. To re-write it in a “useable” form we will apply partial fractions as follows

\[
\frac{6}{(s^2 + 1)(s^2 + 4)} = \frac{A}{s^2 + 1} + \frac{B}{s^2 + 4}
\]
Adding the two terms on the right hand side requires a common denominator

\[
\frac{A}{s^2 + 1} + \frac{B}{s^2 + 4} = \frac{A(s^2 + 4) + B(s^2 + 1)}{(s^2 + 1)(s^2 + 4)} = \frac{As^2 + 4A + Bs^2 + B}{(s^2 + 1)(s^2 + 4)}.
\]

Finally, we match like terms (those with \(s^2\) and those that are constant) in the numerator with the original numerator:

\[
6 = As^2 + 4A + Bs^2 + B
\]

to get two equations which will determine \(A\) and \(B\)

\[
6 = 4A + B \\
0 = A + B
\]

This results in \(A = 2\) and \(B = -2\).
All of this allows us to rewrite the last term in the right hand side of Equation (1) as

\[
\frac{6}{(s^2 + 1)(s^2 + 4)} = 2 \frac{1}{s^2 + 1} - 2 \frac{1}{s^2 + 4}.
\]

This brings us to the last step, which is to invert the Laplace transform on \( Y(s) \) to get \( y(t) = \mathcal{L}^{-1}\{Y(s)\} \).

Since \( Y(s) \) can be simplified to

\[
Y(s) = \frac{2s}{s^2 + 1} + \frac{1}{s^2 + 1} + 2 \frac{1}{s^2 + 1} - 2 \frac{1}{s^2 + 4}
\]

\[
= 2 \left( \frac{s}{s^2 + 1} \right) + 3 \left( \frac{1}{s^2 + 1} \right) - \left( \frac{2}{s^2 + 4} \right)
\]

where each term in parenthesis is in Table 6.2.1, then

The solution to the IVP is

\[
y(t) = 2 \cos(t) + 3 \sin(t) - \sin(2t).
\]
A plot of the solution is shown below.

\[ 1 y'' + 0 y' + 1 y = 3\sin(2t) \]

roots = 0 + 1i, 0 − 1i, \( y(0) = 2, y'(0) = 1 \)
18. Find the solution of the initial value problem

\[ y^{(4)} - y = 0, \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 1, \quad y'''(0) = 0 \]

using the method of Laplace Transforms.

Sol:

- **Transform both sides of the ODE** which results in

  \[ s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) - Y(s) = 0. \]

- **Substitute in the initial conditions** to get

  \[ s^4 Y(s) - s^3 y(0) - s - Y(s) = 0. \]

- **Solve for Y(s)**

  \[ Y(s) = \frac{s^3 + s}{s^4 - 1}. \]
• **Re-write it in a “useable” form to prepare for an inverse Laplace transform**

\[ Y(s) = \frac{s^3 + s}{s^4 - 1} = \frac{s(s^2 + 1)}{(s^2 - 1)(s^2 + 1)} = \frac{s}{s^2 - 1}. \]

• **Invert the Laplace transform**

\[ y(t) = \cosh(t). \]