1. Prove one (551: two) of the following statements

   (a) If $A$ is an $m \times m$ hermetian matrix, then $x^*Ax$ is real, $\forall x \in \mathbb{C}^m$.

   (b) If $A$ is hermetian and $QTQ^*$ is a Schur factorization, then $T$ is diagonal.

   (c) Show that $A$ has real eigenvalues and orthogonal eigenvectors if and only if $A$ is hermitian.

2. Suppose $QTQ^*$ is a Schur factorization of $A$, show that the eigenvalues of $A$ are the diagonal elements of $T$.

3. Compute the LU factorization of the matrix

\[
A = \begin{bmatrix}
1 & 1/2 & 1/3 \\
1/2 & 1/3 & 1/4 \\
1/3 & 1/4 & 1/5
\end{bmatrix}
\]

4. 451: Let $A = R^*R$ be a Cholesky factorization. Is $R^{-1}R^{-*}$ a Cholesky factorization of $A^{-1}$? If so, justify your answer; if not, what is?

5. Compute the eigenvalue decomposition of the matrix

\[
A = \begin{bmatrix}
2 & 1 \\
1 & 2
\end{bmatrix}
\]

6. Compute the SVD of the matrix

\[
A = \begin{bmatrix}
2 & 1 \\
-1 & -2
\end{bmatrix}
\]

7. Consider an overdetermined system $Ax = b$.

   (a) Explain the steps involved in solving the system via the normal equations and Cholesky factorization.

   (b) Give an estimate of the total asymptotic operation count required to solve the least squares problem using this approach. (Also include cost of forming the coefficient matrix of the normal equations.)

   (c) Explain roughly why this approach is bad and why an alternative is better.
8. Consider the system of linear equations \( Ax = b \). Let \( \delta x \) be the perturbation in \( x \) induced by a perturbation \( \delta b \) in the vector \( b \). Prove that

\[
\frac{||\delta x||}{||x||} \leq \kappa(A) \frac{||\delta b||}{||b||},
\]

where \( \kappa(A) \) is the condition number of \( A \).

9. Give a total asymptotic operation count of one step of the QR algorithm (without shifts) for finding eigenvalues.

10. Let \( R^{(k)} := R^{(k)} R^{(k-1)} \ldots, R^{(1)} \) and \( Q^{(k)} := Q^{(1)} Q^{(2)} \ldots, Q^{(k)} \), where each \( R^{(i)} \) and \( Q^{(i)} \) are those defined by the QR Algorithm. Show that \( A^k = Q^{(k)} R^{(k)} \), i.e., the QR Algorithm computes the QR factorization of the \( k \)th power of \( A \). (Hint: \( A^{(k)} = (Q^{(k)})^* A Q^{(k)} \).)