MTH 656

Uncertainty Quantification

Homework 2

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1. Consider the linearized predator-prey model (around a non-zero equilibrium) given by

$$\begin{aligned} \dot{\vec{w}} + A\vec{w} &= 0, \qquad 0 < t < T \\ \vec{w}(0) &= \vec{w}_I \end{aligned}$$

where

$$A = \begin{bmatrix} 0 & -\alpha \\ \beta & 0 \end{bmatrix}$$

and $\vec{w} = [u, v]^T$ represents the deviation from equilibrium. The eigenvalues are $\lambda = \pm \mathrm{i}\sqrt{\alpha\beta}$, which implies that the equilibrium is a stable cycle

Now assume that α and β are parameters estimated using measurements which are subject to a relative error with noise level r and bias m, i.e., $\hat{\alpha} = (r\xi + m)\alpha$ and $\hat{\beta} = (r\xi + m)\beta$, with $\xi \sim U[-1, 1]$. The random ODE incorporating uncertainty is

$$\dot{\vec{w}} + (r\xi + m)A\vec{w} = 0.$$

(a) Expanding each term of $\vec{w} = [u, v]^T$ with a Polynomial Chaos expansion of degree two and substitute into the RODE in order to show by example that the modal equations are

$$\dot{\vec{w}}_N + A \otimes (rM_N + mI_N)\vec{w}_N = \vec{0}.$$

(b) What are the eigenvalues of the modal system for $N = 2$?

(c) Use `ode45` or similar to simulate the modal equations and plot the expected values with the following parameter values: $u_I = v_I = 100$, $T = 250$, $\alpha = \beta = 0.5$, and the following cases

   i. $r = 0$, $m = 1$

   ii. $r = 0.01$, $m = 1.1$

For each case, comment on qualitative behavior of solutions, in particular, are solutions stable oscillations, or decaying?

(d) Do the simulations agree with the theory that says if $r < m$ then the sign of the real part of the eigenvalues is preserved? Do the simulations match what you would expect given the eigenvalues of the model system found in part (1b)?

(e) (Optional) Simulate the RODE directly using 10 random samples of ξ with $r = 0.01$, $m = 1.1$ and plot each u on the same graph, and separately plot each v on the same graph. Comment on whether each is a stable cycle. Then compute the sample mean of 1000 random realizations of the solutions and compare these plots to the Stochastic Galerkin approximation. Is the sample mean a stable cycle?

2. Consider the non-linear initial value problem with random input

$$\begin{aligned} \frac{d}{dt}y(t,\xi) &= y(1 - y/K) \\ y(0,\xi) &= 1 + \xi, \end{aligned}$$

where $\xi \sim \mathcal{N}(0,1)$ and $K > 0$ is a constant. Note that this is a logistic (population) model with carrying capacity K. Assume $K = 1$ for simplicity. Thus $y = 1$ is a stable equilibrium solution.

(a) (Optional) Use `ode45` (or similar) to plot 10 random samples of the solution to the RODE with $0 \leq t \leq 10$ on the same plot. Use `axis([0 10 -5 5])`.

(b) Derive the Hermite Chaos system of degree 1, including modal equations and initial conditions.

(c) Either use `ode45` (or similar) to simulate the y_0 approximation of the expected value of y, or use the fact that in this case it just so happens that $y_0 = y_1$ to solve the modal equations analytically (basically another logistic but with $K = 1/2$). Comment on the value of the stable equilibrium of the modal system compared to what you may have expected given the stable equilibrium of the original system.