MTH 656

Uncertainty Quantification

Homework 4

Euler-Maruyama

Consider the nonlinear stochastic (logistic) differential equation

$$dX(t) = rX(t)(K - X(t))dt + \beta X(t)dW(t), 0 < t \leq T \tag{1}$$

with $X(0) = X_0 \geq 0$. This model is often used for modeling population growth with K representing environmental carrying capacity, growth rate r , and β being a measure of the amount of noise in the system. It can be shown that the unique (strong) solution is

$$X(t) = \frac{\exp\left((rK - \beta^2/2)t + \beta W(t)\right)}{1/X_0 + r\int_0^t \exp\left((rK - \beta^2/2)s + \beta W(s)\right)ds}. \tag{2}$$

Download matlab file `milstein.m` from the course website (basically the same as Higham's `milstrong.m` but with a plot of sample paths added). This code uses $r = 2$, $K = 1$, $\beta = 0.25$ and $X_0 = 0.5$.

1. Modify `milstein.m` to perform the Euler-Maruyama method on (1). Be sure to save the file with a different filename, e.g., `emlogistic.m`. Also, change the titles of plots to EM to avoid confusion. Comment of the accuracy (strong convergence rate) of the method as compared to Milstein for this problem. What about when β is increased to 0.5 or 0.75? Why might there be a difference

2. Modify `milstein.m` to perform the Stochastic Backward Euler method (i.e., Stochastic theta method with $\theta = 1$) on (1). Be sure to save the file with a different filename, e.g., `belogistic.m`. Also, change the titles of plots to BE to avoid confusion. Comment of the accuracy (strong convergence rate) of the method as compared to Milstein for this problem. What about when β is increased to 0.5 or 0.75? (Hint: it is helpful to write out the method by hand and solve the quadratic equation exactly for the positive root in order to determine the update step.)

3. (Optional) Repeat above for $\theta = \frac{1}{2}$.

4. (Optional) Modify `stint.m` from `http://personal.strath.ac.uk/d.j.higham/algfiles.html` to simulate the exact solution (2).

5. (Optional) Modify `stab.m` from `http://personal.strath.ac.uk/d.j.higham/algfiles.html` to examine the stability of Backward Euler. In particular, test

   (a) mean square convergence for parameter values

      i. $\lambda = -3$ and $\mu^2 = 3$

      ii. $\lambda = 3$ and $\mu^2 = 3$

   (b) asymptotic convergence for parameter values

      i. $\lambda = 0$ and $\mu^2 = 6$

      ii. $\lambda = 3$ and $\mu^2 = 3$

   in each case with $\Delta t \in \{1, \frac{1}{2}, \frac{1}{4}\}$. Comment for each case on whether the SDE is stable (for that particular mode of convergence and set of parameter values) as well as which time steps produce simulations that agree with the stability of the SDE.