Finite Difference, Finite Element and Finite Volume Methods for the Numerical Solution of PDEs

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Math Modeling and Simulation of Physical Processes

- Define the physical problem
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- Create a mathematical (PDE) model
  - Systems of PDEs, ODEs, algebraic equations
  - Define Initial and or boundary conditions to get a well-posed problem
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• Create a Discrete (Numerical) Model
  • Discretize the domain → generate the grid → obtain discrete model
  • Solve the discrete system
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  - Discretize the domain → generate the grid → obtain discrete model
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- Analyse Errors in the discrete system
  - Consistency, stability and convergence analysis
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- Partial Differential Equations (PDEs)
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- Conservation Laws: Integral and Differential Forms
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- Conservation Laws: Integral and Differential Forms
- Classification of PDEs: Elliptic, parabolic and Hyperbolic
- Finite difference methods
- Analysis of Numerical Schemes: Consistency, Stability, Convergence
- Finite Volume and Finite element methods
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- Conservation Laws: Integral and Differential Forms
- Classification of PDEs: Elliptic, parabolic and Hyperbolic
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- Analysis of Numerical Schemes: Consistency, Stability, Convergence
- Finite Volume and Finite element methods
- Iterative Methods for large sparse linear systems
Partial Differential Equations

- PDEs are mathematical models of continuous physical phenomenon in which a dependent variable, say \( u \), is a function of more than one independent variable, say \( t \) (time), and \( x \) (eg. spatial position).

Boundary conditions, i.e., conditions on the (finite) boundary of the domain and/or initial conditions (for transient problems) are required to obtain a well-posed problem.
Partial Differential Equations

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- PDEs derived by applying a physical principle such as conservation of mass, momentum or energy. These equations, governing the kinematic and mechanical behaviour of general bodies are referred to as Conservation Laws. These laws can be written in either the strong of differential form or an integral form.
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PDEs (continued)

- For simplicity, we will deal only with single PDEs (as opposed to systems of several PDEs) with only two independent variables,
  - either two space variables, denoted by $x$ and $y$, or
  - one space variable denoted by $x$ and one time variable denoted by $t$
PDEs (continued)

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  - either two space variables, denoted by $x$ and $y$, or
  - one space variable denoted by $x$ and one time variable denoted by $t$
- Partial derivatives with respect to independent variables are denoted by subscripts, for example
  - $u_t = \frac{\partial u}{\partial t}$
  - $u_{xy} = \frac{\partial^2 u}{\partial x \partial y}$
Well Posed Problems

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- Boundary conditions, i.e., conditions on the (finite) boundary of the domain and/or initial conditions (for transient problems) are required to obtain a *well posed problem*.

- Properties of a well posed problem:
  - Solution exists
  - Solution is unique
  - Solution depends continuously on the data
Classifications of PDEs

• The **Order** of a PDE = the highest-order partial derivative appearing in it. For example,
  • The **advection equation** \( u_t + u_x = 0 \) is a first order PDE.
  • The **Heat equation** \( u_t = u_{xx} \) is a second order PDE.
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  - The **Heat equation** $u_t = u_{xx}$ is a second order PDE.
- A PDE is **linear** if the coefficients of the partial derivatives are not functions of $u$, for example
  - The **advection equation** $u_t + u_x = 0$ is a linear PDE.
  - The **Burgers equation** $u_t + uu_x = 0$ is a nonlinear PDE.
Classifications of PDEs (continued)

Second-order linear PDEs of general form

\[ au_{xx} + bu_{xy} + cu_{yy} + du_x + e u_y + f u + g = 0 \]

are classified based on the value of the discriminant \( b^2 - 4ac \):

- \( b^2 - 4ac > 0 \): hyperbolic
  - e.g., wave equation: \( u_{tt} - u_{xx} = 0 \)
  - Hyperbolic PDEs describe time-dependent, conservative physical processes, such as convection, that are not evolving toward steady state.
Classifications of PDEs (continued)

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- \( b^2 - 4ac = 0 \): parabolic
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  - Parabolic PDEs describe time-dependent dissipative physical processes, such as diffusion, that are evolving toward steady state.
Classifications of PDEs (continued)

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  - Parabolic PDEs describe time-dependent dissipative physical processes, such as diffusion, that are evolving toward steady state.

- \( b^2 - 4ac < 0 \): elliptic
  - e.g., Laplace equation: \( u_{xx} + u_{yy} = 0 \)
  - Elliptic PDEs describe processes that have already reached steady states, and hence are time-independent.
Parabolic PDEs: Initial-Boundary value problems

- Example: One dimensional (in space) Heat Equation for $u = u(t, x)$

\[ u_t = \kappa u_{xx}, \quad 0 \leq x \leq L, \quad t \geq 0 \]
Parabolic PDEs: Initial-Boundary value problems

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• with
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Elliptic PDEs: Boundary value problems

- Example: Model of steady heat conduction in a two dimensional (in space) domain, governed by the Laplace equation for the temperature

\[ T = T(x, y) \]

\[ T_{xx} + T_{yy} = 0, \ 0 \leq x \leq W, 0 \leq y \leq H \]
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• with boundary conditions
  • $T(x, 0) = T_1$, $T(x, H) = T_3$
  • $T(0, y) = T_4$, $T(W, y) = T_2$
Elliptic PDEs: Boundary value problems

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Hyperbolic PDEs: Initial-Boundary value problems

- Example: One-dimensional (in space) wave equation for $u = u(t, x)$

$$u_{tt} = c^2 u_{xx}, \; 0 \leq x \leq L, \; t \geq 0$$
Hyperbolic PDEs: Initial-Boundary value problems

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![Diagram](image_url)
Finite Difference Methods (FDM): Discretization

- Suppose that we are solving for \( u = u(t, x) \) on the domain \( \Omega = [0, T] \times [0, L] \). We discretize the domain \( \Omega \) by partitioning the spatial interval \([0, L]\) into \( m + 2 \) grid points \( x_0, x_1, \ldots, x_m, x_{m+1} = L \), such that

\[
\Delta x_j = x_{j+1} - x_j, \quad j = 0, 1, 2, \ldots m
\]

In the case that the \( m + 2 \) spatial points \( x_j \) are equally spaced, we have

\[
\Delta x = \Delta x_j, \quad \forall j
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In the case that the \( m + 2 \) spatial points \( x_j \) are equally spaced, we have

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- We similarly discretize the temporal domain \([0, T]\) into discrete time levels \( t_k \) with time step \( k = \Delta t \). 
Finite Difference Methods: Discretization

- The numerical solution to the PDE is an approximation to the exact solution that is obtained using a discrete representation to the PDE at the grid points $x_j$ in the discrete spatial mesh at every time level $t_k$. Let us denote this numerical solution as $U$ such that

$$U^n_j \approx u(t_k, x_j)$$
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- Thus, the numerical solution is a collection of finite values,

$$U^n = [U^n_1, U^n_2, \ldots, U^n_m]$$

at each time level $t_n$. 
Finite Difference Methods: Discretization

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- Thus, the numerical solution is a collection of finite values,

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at each time level $t_n$.

- The boundary conditions determine the values of $U^n_0$ and $U^n_{m+1}$ for all $n$. The initial conditions determine the values of $U^0$ at each spatial grid point.
Finite Difference Methods (continued)

- Recall the definition of the derivative from introductory Calculus:

\[ u_x(x_j) = \lim_{h \to 0} \frac{u(x_j + h) - u(x_j)}{h} \]

\[ = \lim_{h \to 0} \frac{u(x_j) - u(x_j - h)}{h} \]

\[ = \lim_{h \to 0} \frac{u(x_j + h) - u(x_j - h)}{2h} \]
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$$= \lim_{h \to 0} \frac{u(x_j + h) - u(x_j - h)}{2h}$$

• We use these formula with a small finite value of $h = \Delta x$, i.e., we approximate

$$u_x(x_j) \approx \frac{u(x_j + h) - u(x_j)}{h} \quad \text{(Forward difference)}$$

$$\approx \frac{u(x_j) - u(x_j - h)}{h} \quad \text{(Backward difference)}$$

$$\approx \frac{u(x_j + h) - u(x_j - h)}{2h} \quad \text{(Centered difference)}$$
Error in FDM: Local Truncation Error

- The local truncation error (LTE) is the error that results by substituting the exact solution into the finite difference formula.
Error in FDM: Local Truncation Error

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• Errors in the approximations to the derivative are calculated using Taylor approximations around a grid point $x_j$. For example,
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- The **local truncation error** (LTE) is the error that results by substituting the exact solution into the finite difference formula.
- Errors in the approximations to the derivative are calculated using Taylor approximations around a grid point \( x_j \). For example,

\[
\begin{align*}
    u(x_{j+1}) &= u(x_j + \Delta x) \\
    &= u(x_j) + u_x(x_j) \Delta x + u_{xx}(x_j) \frac{(\Delta x)^2}{2} + O((\Delta x)^3)
\end{align*}
\]
Error in FDM: Local Truncation Error

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$$u(x_{j+1}) = u(x_j + \Delta x) = u(x_j) + u_x(x_j)\Delta x + u_{xx}(x_j)\frac{(\Delta x)^2}{2} + O((\Delta x)^3)$$

- Thus,

$$u_x(x_j) = \frac{u(x_{j+1}) - u(x_j)}{\Delta x} + u_{xx}(x_j)\frac{\Delta x}{2} + O((\Delta x)^2)$$
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$$= u(x_j) + u_x(x_j)\Delta x + u_{xx}(x_j)\frac{(\Delta x)^2}{2} + O((\Delta x)^3)$$

- Thus,

$$u_x(x_j) = \frac{u(x_{j+1}) - u(x_j)}{\Delta x} + u_{xx}(x_j)\frac{\Delta x}{2} + O((\Delta x)^2)$$

- The forward difference is a first order accurate approximation to the partial derivative $u_x$ at $x_j$ and the LTE is $O(\Delta x)$.
Error in FDM: LTE

- The backward difference is a first order accurate approximation to the partial derivative $u_x$ at $x_j$ and the LTE is $O(\Delta x)$.
Error in FDM: LTE

- The backward difference is a first order accurate approximation to the partial derivative $u_x$ at $x_j$ and the LTE is $O(\Delta x)$.

- The centered difference is a second order accurate approximation to the partial derivative $u_x$ at $x_j$ and the LTE is $O((\Delta x)^2)$.
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- The backward difference is a first order accurate approximation to the partial derivative $u_x$ at $x_j$ and the LTE is $O(\Delta x)$.

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- Note that the LTE in all these approximations goes to zero as $\Delta x$ goes to zero.
FDM for Parabolic PDEs: The Heat Equation

Consider the initial-boundary value problem for the heat equation

\[ u_t = \kappa u_{xx}, \quad 0 \leq x \leq 1, \quad t \geq 0 \]

\[ u(0, x) = f(x), \quad \text{Initial Condition} \]

\[ u(t, 0) = \alpha, \quad \text{Boundary Condition at } x = 0 \]

\[ u(t, 1) = \beta, \quad \text{Boundary Condition at } x = 1 \]
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\[ u(t, 1) = \beta, \quad \text{Boundary Condition at } x = 1 \]

- Discretize the spatial domain \([0, 1]\) into \(m + 2\) grid points using a uniform mesh step size \(\Delta x = 1/(m + 1)\). Denote the spatial grid points by \(x_j, j = 0, 1, \ldots, m + 1\).
FDM for Parabolic PDEs: The Heat Equation

- Similarly discretize the temporal domain into temporal grid points $t_k = k\Delta t$ for suitably chosen time step $\Delta t$. 
FDM for Parabolic PDEs: The Heat Equation

- Similarly discretize the temporal domain into temporal grid points $t_k = k\Delta t$ for suitably chosen time step $\Delta t$.

- Denote the approximate solution at the grid point $(t_k, x_j)$ as $U^k_j$.

\[
\alpha = u^k_0 \quad u^k_1 \quad u^k_2 \quad \ldots \quad u^k_{j-1} \quad u^k_j \quad u^k_{j+1} \quad \ldots \quad u^k_m \quad u^k_{m+1} = \beta
\]

\[
0 = x_0 \quad x_1 \quad x_2 \quad \ldots \quad x_{j-1} \quad x_j \quad x_{j+1} \quad \ldots \quad x_m \quad x_{m+1} = 1
\]
FDM for Parabolic PDEs: The Heat Equation

- Similarly discretize the temporal domain into temporal grid points \( t_k = k\Delta t \) for suitably chosen time step \( \Delta t \).
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\[
\alpha = u_0^k \quad u_1^k \quad u_2^k \quad \ldots \quad u_{j-1}^k \quad u_j^k \quad u_{j+1}^k \quad \ldots \quad u_m^k \quad u_{m+1}^k = \beta \quad (t_k = k\Delta t)
\]

- The space-time grid can be represented as
FDM for Parabolic PDEs: The Heat Equation

- Replace $u_t$ by a forward difference in time and $u_{xx}$ by a central difference in space to obtain the explicit FDM
FDM for Parabolic PDEs: The Heat Equation

- Replace $u_t$ by a forward difference in time and $u_{xx}$ by a central difference in space to obtain the explicit FDM

\[ \frac{U_j^{k+1} - U_j^k}{\Delta t} = \kappa \frac{U_{j+1}^k - 2U_j^k + U_{j-1}^k}{(\Delta x)^2} \]

\[ \Rightarrow U_j^{k+1} = U_j^k + \frac{\kappa \Delta t}{(\Delta x)^2} (U_{j+1}^k - 2U_j^k + U_{j-1}^k), \quad j = 1, 2, \ldots m \]
FDM for Parabolic PDEs: The Heat Equation

- Replace $u_t$ by a forward difference in time and $u_{xx}$ by a central difference in space to obtain the explicit FDM

$$\frac{U_j^{k+1} - U_j^k}{\Delta t} = \kappa \frac{U_{j+1}^k - 2U_j^k + U_{j-1}^k}{(\Delta x)^2}$$

$$\implies U_j^{k+1} = U_j^k + \frac{\kappa \Delta t}{(\Delta x)^2} \left( U_{j+1}^k - 2U_j^k + U_{j-1}^k \right), \quad j = 1, 2, \ldots m$$

- Associated to this scheme is a Computational Stencil

\[ \begin{array}{c}
 k + 1 \\
 k \\
 k - 1 \\
 j - 1 & j & j + 1 \\
\end{array} \]
FDM for Parabolic PDEs: The Heat Equation

- This is an explicit FDM for the heat equation: Solution at time level $k + 1$ is determined by solution at previous time levels only.

From Boundary conditions:

$U_{k+1}^{0} = \text{and } U_{k+1}^{m} = \text{for all values of } k.$

From Initial condition:

$U_{0}^{j} = f(x^j)$ for all values of $j.$

The local truncation error is $O(t) + O((x)^2).$

Scheme is first order accurate in time and second order accurate in space.
FDM for Parabolic PDEs: The Heat Equation

- This is an explicit FDM for the heat equation: Solution at time level \( k + 1 \) is determined by solution at previous time levels only.
- We note that
  - From Boundary conditions: \( U_0^k = \alpha \) and \( U_{m+1}^k = \beta \) for all values of \( k \).
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  Scheme is first order accurate in time and second order accurate in space

- How do we choose the values of \( \Delta t \) and \( \Delta x \)??
FDM for Parabolic PDEs: The Heat Equation

- Initial condition has a discontinuous derivative at $x = 0.5$. 
FDM for Parabolic PDEs: The Heat Equation

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However, we see a rapid smoothing effect of this initial discontinuity as time evolves. In general high frequencies get rapidly damped as compared to low frequencies. We say that the heat equation is stiff. In the above we assumed that the value of $r = \frac{1}{2}$. Here $x = 0$, and $t = 10$.
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- In the above we assumed that the value of

$$r = \frac{\kappa \Delta t}{\Delta x} \leq \frac{1}{2}$$

Here $\Delta x = 0.01$ and $\Delta t = 10^{-5}$
FDM for Parabolic PDEs: The Heat Equation

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- However, we see a rapid smoothing effect of this initial discontinuity as time evolves.
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- In the above we assumed that the value of

\[
r = \frac{\kappa \Delta t}{\Delta x} \leq \frac{1}{2}
\]

Here \( \Delta x = 0.01 \) and \( \Delta t = 10^{-5} \)

- What happens if this value is greater than 1/2?
FDM for Parabolic PDEs: The Heat Equation

We see unstable behavior of the numerical solution! The numerical solution does not stay bounded.
We see **unstable** behavior of the numerical solution! The numerical solution does not stay bounded.

Thus, $\Delta t$ and $\Delta x$ cannot be chosen arbitrarily. They have to satisfy a **stability condition**.
Implicit FDM for Parabolic PDEs: The Heat Equation

- Replace $u_t$ by a forward difference in time and $u_{xx}$ by a central difference in space to obtain the Implicit FDM
Implicit FDM for Parabolic PDEs: The Heat Equation

- Replace $u_t$ by a forward difference in time and $u_{xx}$ by a central difference in space to obtain the Implicit FDM

\[
\frac{U_j^{k+1} - U_j^k}{\Delta t} = \kappa \frac{U_{j+1}^{k+1} - 2U_j^{k+1} + U_{j-1}^{k+1}}{(\Delta x)^2}
\]

\[\implies U_j^{k+1} = U_j^k + \frac{\kappa \Delta t}{(\Delta x)^2} (U_{j+1}^{k+1} - 2U_j^{k+1} + U_{j-1}^{k+1}), \ j = 1, 2, \ldots m\]
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\[\Rightarrow U_j^{k+1} = U_j^k + \frac{\kappa \Delta t}{\Delta x^2} (U_{j+1}^{k+1} - 2U_j^{k+1} + U_{j-1}^{k+1}), \quad j = 1, 2, \ldots m\]

- Associated to this scheme is a Computational Stencil

\[
\begin{array}{cccccccc}
& & & & k + 1 & & & \\
& & & & \uparrow & & & \\
& & & \bullet & & \bullet & & \bullet \\
& & \bullet & & \bullet & & \bullet & \\
k & & \bullet & & \bullet & & \bullet & \\
& k - 1 & & \bullet & & \bullet & & \bullet & \\
& j - 1 & & j & & j + 1 & & \\
\end{array}
\]
We see stable behavior of the numerical solution! The numerical solution remains bounded even when $r > 1/2$. 

\[ \text{Initial function} \]

\[ \text{Exact Solution} \]

\[ \text{dx}=0.01, \text{dt}=0.01, r=100 \]
FDM for Parabolic PDEs: The Heat Equation

- We see **stable** behavior of the numerical solution! The numerical solution remains bounded even when $r > 1/2$.
- Thus, $\Delta t$ and $\Delta x$ can be chosen to have the same order of magnitude.
• We see stable behavior of the numerical solution! The numerical solution remains bounded even when \( r > \frac{1}{2} \).
• Thus, \( \Delta t \) and \( \Delta x \) can be chosen to have the same order of magnitude.
• The implicit FDM is unconditionally stable.
Crank-Nicolson for The Heat Equation

- Replace $u_t$ by a forward difference in time and $u_{xx}$ by a central difference in space to obtain the Implicit FDM.
Crank-Nicolson for The Heat Equation

- Replace $u_t$ by a forward difference in time and $u_{xx}$ by a central difference in space to obtain the \textbf{Implicit FDM}

\[
\frac{U_j^{k+1} - U_j^k}{\Delta t} = \frac{\kappa}{2} \left( \frac{U_{j+1}^{k+1} - 2U_j^{k+1} + U_{j-1}^{k+1}}{(\Delta x)^2} \right) + \frac{\kappa}{2} \left( \frac{U_{j+1}^k - 2U_j^k + U_{j-1}^k}{(\Delta x)^2} \right)
\]

\[\implies U_j^{k+1} = U_j^k + \frac{\kappa \Delta t}{2(\Delta x)^2} \left( U_{j+1}^{k+1} - 2U_j^{k+1} + U_{j-1}^{k+1} + U_{j+1}^k - 2U_j^k + U_{j-1}^k \right), \quad j = 1, 2, \ldots m\]
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$$\Rightarrow U_j^{k+1} = U_j^k + \frac{\kappa \Delta t}{2(\Delta x)^2} \left( U_{j+1}^k - 2U_j^k + U_{j-1}^k + U_{j+1}^{k+1} - 2U_j^{k+1} + U_{j-1}^{k+1} \right),$$

$j = 1, 2, \ldots m$

- Associated to this scheme is a Computational Stencil

- This method is unconditionally stable,
- and second-order accurate in time
- However a system of equations must be solved at each time step.
First vs Second Order Accuracy

\[ \log_2(\Delta x) \]

\[ \log_2(\text{LTE}) \]

Slope = 1: First order accurate

Slope = 2: Second Order accurate
Method of Lines (MOL) Discretization

- Another way of solving time dependent PDEs numerically is to discretize in space but not in time.

This results in a large coupled system of ODEs which we can then solve using numerical methods developed for ODEs, such as Forward and Backward Euler method, Trapeziolal methods, Runge-Kutta methods etc.

The method of lines approach can be used to analyze the stability of the numerical method for the PDE by analyzing the eigenvalues of the matrix of the resulting system of ODEs using the ideas of absolute stability for ODEs.
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Analysis of FDM

- Consistency implies that the local truncation error goes to zero as $\Delta x$ and $\Delta t$ approach zero. This is usually proved by invoking Taylor’s theorem.
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- Stability implies that the numerical solution remains bounded at any given time $t$. Stability is harder to prove than consistency. Stability can be proven using either
  - Eigenvalue analysis of the matrix representation of the FDM.
  - Fourier analysis on the grid (von Neumann analysis)
  - Computing the domain of dependence of the numerical method.
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  - Computing the domain of dependence of the numerical method.

- **Lax-Equivalence Theorem**: A consistent approximation to a well-posed problem is convergent if and only if it is stable.
Von Neumann Analysis for Time-dependent Problems

- Example: The analytical solutions of the heat equation $u_t - \kappa u_{xx} = 0$
  can be found in the form

$$u(t, x) = \sum_{-\infty}^{\infty} e^{\beta_m t} e^{i\alpha_m x}$$

with $\beta_m + \kappa \alpha_m^2 = 0$. Here $e^{i\alpha_m x} = \cos(\alpha_m x) + i \sin(\alpha_m x)$.
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**Von Neumann Analysis for Time-dependent Problems**

- Example: The analytical solutions of the heat equation \( u_t - \kappa u_{xx} = 0 \) can be found in the form

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- In the discrete case we assume that \( U^k_j = G^k e^{i\alpha_m j \Delta x} \) with \( G = e^{\beta_m \Delta t} \). Any growth in the solution will be due to the presence of terms involving \( G \).

- Thus requiring that this amplification factor \( G \) is bounded by one as \( k \to \infty \) gives rise to a relation between \( \Delta t \) and \( \Delta x \) called the von Neumann stability condition.
FDM for Hyperbolic PDEs: The Advection Equation

- Consider the initial value problem for the Advection equation

\[ u_t + a u_x = 0, \quad 0 \leq x \leq 1, \quad t \geq 0 \]

\[ u(0, x) = f(x), \quad \text{Initial Condition} \]
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- Information propagates along characteristics
FDM for the Advection Equation

- Replace $u_t$ by a forward difference in time and $u_x$ by a backward difference in space to obtain the explicit FDM.
FDM for the Advection Equation

- Replace $u_t$ by a forward difference in time and $u_x$ by a backward difference in space to obtain the explicit FDM

\[
\frac{U_{j}^{k+1} - U_{j}^{k}}{\Delta t} + a \frac{U_{j+1}^{k} - U_{j}^{k}}{\Delta x} = 0
\]

\[
\Rightarrow U_{j}^{k+1} = U_{j}^{k} + \frac{a \Delta t}{\Delta x} (U_{j}^{k} - U_{j-1}^{k}) \quad j = 1, 2, \ldots, m
\]
FDM for the Advection Equation

- Replace $u_t$ by a forward difference in time and $u_x$ by a backward difference in space to obtain the **explicit FDM**

\[
\frac{U_{j}^{k+1} - U_{j}^{k}}{\Delta t} + a \frac{U_{j+1}^{k+1} - U_{j}^{k+1}}{\Delta x} = 0
\]

\[
\implies U_{j}^{k+1} = U_{j}^{k} + \frac{a \Delta t}{\Delta x} (U_{j}^{k} - U_{j-1}^{k}), \quad j = 1, 2, \ldots m
\]

- Associated to this scheme is a **Computational Stencil**

\[
\begin{array}{c}
  k + 1 \\
  k \\
  k - 1 \\
  j - 1 \quad j \quad j + 1
\end{array}
\]

- Scheme is explicit
- First order accurate is time and space
- $\Delta t$ and $\Delta x$ are related through the **Courant number**

\[
\nu = \frac{a \Delta t}{\Delta x}
\]
Courant Friedrich Lewy (CFL) Condition

- **The CFL Condition**: For stability, at each mesh point, the Domain of dependence of the PDE must lie within the domain of dependence of the numerical scheme.
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- CFL is a necessary condition for stability of explicit FDM applied to Hyperbolic PDEs. It is not a sufficient condition.
Courant Friedrich Lewy (CFL) Condition

- **The CFL Condition**: For stability, at each mesh point, the Domain of dependence of the PDE must lie within the domain of dependence of the numerical scheme.

- CFL is a necessary condition for stability of explicit FDM applied to Hyperbolic PDEs. It is not a sufficient condition.

- For the advection equation CFL condition for stability is $|\nu| \leq 1$. i.e.,

  $$\Delta t \leq \frac{\Delta x}{|a|}.$$
Elliptic PDEs: Laplace Equation

- Time-independent problems
Elliptic PDEs: Laplace Equation

- Time-independent problems
- Consider the Boundary value problem for Laplace equation in two spatial dimensions

\[ u_{xx} + u_{yy} = 0, \; 0 \leq x \leq 1, \; 0 \leq y \leq 1 \]

with boundary conditions prescribed as shown below
FDM for Elliptic PDEs: Laplace Equation

- Discretize the mesh using uniform mesh step in both the $x$ and $y$ directions as below.

\[
\begin{align*}
  y_{m+1} &= 1 \\
  y_{m} &= 1 \\
  y_{m-1} &= \Delta y \\
  \vdots &= \Delta x \\
  y_{2} &= \Delta x \\
  y_{1} &= \Delta x \\
  0 &= y_{0} \\
  0 &= x_{0} \quad x_{1} \quad x_{2} \quad \ldots \quad \ldots \quad x_{j-1} \quad x_{j} \quad x_{j+1} \quad \ldots \quad x_{m} \quad x_{m+1} = 1
\end{align*}
\]
Elliptic PDEs: Laplace Equation

- Replace both the second order derivatives $u_{xx}$ and $u_{yy}$ with centered differences at each grid point $(x_j, y_k)$ to obtain the difference scheme

$$\frac{U_{j+1,k} - 2U_{j,k} + U_{j-1,k}}{(\Delta x)^2} + \frac{U_{j,k+1} - 2U_{j,k} + U_{j,k-1}}{(\Delta y)^2} = 0$$
Elliptic PDEs: Laplace Equation

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$$
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$$

- If $\Delta x = \Delta y$ this becomes

$$
U_{j+1,k} + U_{j-1,k} + U_{j,k+1} + U_{j,k-1} - 4U_{j,k} = 0
$$
Elliptic PDEs: Laplace Equation

- Replace both the second order derivatives \( u_{xx} \) and \( u_{yy} \) with centered differences at each grid point \((x_j, y_k)\) to obtain the difference scheme

\[
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\]

- The Stencil for this FDM is called the **Five-Point Stencil**
Elliptic PDEs: Laplace Equation

- This FDM gives rise to a system of linear equations of the form

\[ AU = b \]

- The right hand side vector \( b \) contains the boundary information.
- The vector \( U \) is the solution vector at the interior grid points.
- The matrix \( A \) is block tridiagonal if ordered in a *natural way*
- The structure of \( A \) depends on the ordering of the grid points.
- This system can be solved by iterative techniques or direct methods such as Gaussian elimination.
Elliptic PDEs: Laplace Equation

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- The structure of \( A \) depends on the ordering of the grid points.
- This system can be solved by iterative techniques or direct methods such as Gaussian elimination.

- When \( m = 2 \), the system \( AU = b \) can be written as

\[
\begin{bmatrix}
-4 & 1 & 1 & 0 \\
1 & -4 & 0 & 1 \\
1 & 0 & -4 & 1 \\
0 & 1 & 1 & -4 \\
\end{bmatrix}
\begin{bmatrix}
U_{1,1} \\
U_{2,1} \\
U_{1,2} \\
U_{2,2} \\
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
1 \\
1 \\
\end{bmatrix}
\]
Discretization of Elliptic PDEs

Each block $(L + 1) \times (L + 1)$
Finite Element Method

Features

- Flexibility
- Complicated geometries
- High-order approximations
- Strong mathematical foundation
Basic Idea

\[ u(x) \approx \hat{u}(x) = \sum_{j=1}^{M} u_j \phi_j(x) \]

- \( \phi_j \) are basis functions
- \( u_j \): \( M \) unknowns; Need \( M \) equations
- Discretizing derivatives results in linear system
1D Elliptic Example

\[-u'' = f, \quad 0 < x < 1\]

\[u(0) = u(1) = 0\]

- For example, elastic cord with fixed ends
- Solution must be twice differentiable
- This is unnecessarily strong if \( f \) is discontinuous
Weak Formulation

Multiply both sides by an arbitrary test function $v$ and integrate

\[
\int_0^1 -u''v\,dx = \int_0^1 f\,vdx
\]

\[
\int_0^1 u'v'\,dx - u'v\big|_0^1 = \int_0^1 f\,vdx.
\]

\[
\int_0^1 u'v'\,dx = \int_0^1 f\,vdx.
\]

Since $v$ was arbitrary, this equation must hold for all $v$ such that the equation makes sense ($v'$ is square integrable), and $v(0) = v(1) = 0$. 

Weak Formulation 2

Find $u \in V$ such that

$$\int_{0}^{1} u' v' \, dx = \int_{0}^{1} f v \, dx \quad \forall v \in V$$

where $V = H_{0}^{1}([0, 1])$.

- Fewer derivatives required for $u$
- If $f$ continuous, same $u$ as strong form
- Infinite possibilities for $v$
- Want to find $u$ on a discrete mesh
Finite-dimensional Subspace

• Let $0 = x_0 < x_1 < \ldots < x_{M+1} = 1$ be a partition of the domain with $h_j = x_j - x_{j-1}$ and $h = \max h_j$. Use the partition to define a finite-dimensional subspace $V_h \subset V$.

• For decreasing $h$, want that functions in $V_h$ can get arbitrarily close to functions in $V$.

• For example, let $V_h$ be piecewise linear (i.e., on each subinterval) functions such that $\hat{v} \in V_h$ is continuous on $[0, 1]$ and $\hat{v}(0) = \hat{v}(1) = 0$.

• We may introduce basis functions $\phi_j(x)$ such that $\phi_j(x_i) = \delta_{ij}$ for $i, j = 0, \ldots, M + 1$.

• Nodes $x_i$ are sometimes denoted $N_i$. 
Basis Functions

\[ u_1 x + u_i x + u_N x = \sum_{i=1}^{N} u_i x \]

For \( u_i \) at each \( x_i \):

\[ u_i(x) = \begin{cases} 1 & \text{for } x_i - 1 < x < x_i \\ 0 & \text{else} \end{cases} \]

\[ \Omega_i = (x_i - 1, x_i) \]
Finite Element Method

Find \( \hat{u} \in V_h \) such that

\[
\int_0^1 \hat{u}' \hat{v}' \, dx = \int_0^1 f \hat{v} \, dx \quad \forall \hat{v} \in V_h.
\]

or

\[
\int_0^1 \hat{u}' \phi_j' \, dx = \int_0^1 f \phi_j \, dx \quad j = 1, \ldots, M.
\]

Note:

- When \( \hat{u} \) and \( \hat{v} \) in same subspace: Galerkin
- If support of \( \phi_i \) is entire space: Spectral
- If \( V_h \) not a subspace of \( V \): Non-conforming
Linear System

- Can represent \( \hat{u} = \sum_{i=1}^{M} \xi_i \phi_i(x) \)
- Find \( \xi_i \) for \( i = 1, \ldots, M \) such that

\[
\int_0^1 \sum_{i=1}^{M} \xi_i \phi_i' \phi_j' \, dx = \int_0^1 f \phi_j \, dx \quad j = 1, \ldots, M.
\]

- Thus if \( A = (a_{ij}) \) with \( a_{ij} = \int_0^1 \phi_i' \phi_j' \, dx \) and \( b = (b_i) \) with \( b_i = \int_0^1 f \phi_i \, dx \), then

\[
A \xi = b
\]
Stiffness Matrix

- The $M \times M$ matrix $A$ is called the stiffness matrix.
- For the piecewise linear basis functions we have chosen it will be tridiagonal.
- For the special case when $h_j \equiv h$ we have

$$
\begin{bmatrix}
2 & -1 & 0 & \cdots & \cdots & 0 \\
-1 & 2 & -1 & \ddots & & \\
0 & -1 & 2 & -1 & \ddots & \\
\vdots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & -1 & 2 & -1 \\
\end{bmatrix}
$$
Compare to FDM

- Note that if Trapezoid rule is used to approximate the right hand side, then $b_i = hf_i$, and therefore the equations determining $\hat{u}$ are

$$\frac{\xi_{i+1} - 2\xi_i + \xi_{i-1}}{h} = hf_i$$

which are exactly the same as FDM.

- The advantage of the FEM formulation is the generality it allows (e.g., uniform $h$ was not required).
Multidimensional Problem

- Let \( \Omega \) be the unit square \((x, y) \in [0, 1] \times [0, 1]\)
- Assume homogeneous Dirichlet boundary conditions
- Then the 2D Possion problem is:
  Find \( u \in V := H^1_0(\Omega) \) such that

\[
-\Delta u = f \quad \text{in} \; \Omega, \\
u = 0 \quad \text{on} \; \partial \Omega.
\]
Variational Formulation

Find $u \in V$ such that

$$a(u, v)_\Omega = (f, v)_\Omega, \quad \forall v \in V,$$

where

$$a(u, v)_\Omega := \int_\Omega \nabla u \cdot \nabla v,$$

$$(f, v)_\Omega := \int_\Omega f v.$$

Note: used Green’s formula and $v = 0$ on $\partial \Omega$. 
\[ P_1 \text{ Finite Element} \]

We introduce a triangulation \( T_h \) of \( \Omega \) into triangles \( K_i \), and a finite dimensional subspace:

\[
V_h := \{ \hat{v} \in H^1_0(\Omega) : \hat{v}|_{K_i} \in P_1(K_i) \}. 
\]

Find \( \hat{u} \in V_h \) such that

\[
a(\hat{u}, \hat{v})_\Omega = (f, \hat{v})_\Omega, \quad \forall \hat{v} \in V_h.
\]
Using Basis Functions

Representing $\hat{u}$ and $\hat{v}$ in terms of nodal basis functions $\{\phi_i\}_{i=1}^{M}$ of $V_h$, i.e., $\phi_j(N_i) = \delta_{ij}$ for $i, j = 0, \ldots, M + 1$, we get the following system of algebraic equations:

$$\sum_{j=1}^{M} \xi_j a(\phi_i, \phi_j) = (f, \phi_i), \quad i = 1, \ldots, M,$$

or in matrix form,

$$A\xi = b,$$

where $\xi_j = \hat{u}(N_j)$, $A_{ij} = a(\phi_i, \phi_j)_{\Omega}$ is SPD, and $b_i = (f, \phi_i)_{\Omega}$. 

Multidimensional Problem

- Let $\Omega$ be the unit cube $\{x, y, z\} \in [0, 1]$
- Assume homogeneous Dirichlet boundary conditions
- Then the 3D Possion problem is:
  Find $u \in V := H^1_0(\Omega)$ such that
  \[-\Delta u = f \quad \text{in } \Omega,\]
  \[u = 0 \quad \text{on } \partial\Omega.\]
Variational Formulation

Find $u \in V$ such that

$$a(u, v)_\Omega = (f, v)_\Omega, \quad \forall v \in V,$$

where

$$a(u, v)_\Omega := \int_{\Omega} \nabla u \cdot \nabla v,$$

$$(f, v)_\Omega := \int_{\Omega} f v.$$

Note: used Green’s formula and $v = 0$ on $\partial \Omega$. 
\( Q_1 \) Finite Element

We introduce a triangulation \( T_h \) of \( \Omega \) into 3-D rectangles \( K_i \), and a finite dimensional subspace:

\[
V_h := \{ \hat{v} \in H^1_0(\Omega) : \hat{v}|_{K_i} \in Q_1(K_i) \}.
\]

Find \( \hat{u} \in V_h \) such that

\[
a(\hat{u}, \hat{v})_\Omega = (f, \hat{v})_\Omega, \quad \forall \hat{v} \in V_h.
\]
Using Basis Functions

Representing \( \hat{u} \) and \( \hat{v} \) in terms of nodal basis functions \( \{ \phi_i \}_{i=1}^{M} \) of \( V_h \), i.e., \( \phi_j(N_i) = \delta_{ij} \) for \( i, j = 0, \ldots, M + 1 \), we get the following system of algebraic equations:

\[
\sum_{j=1}^{M} \xi_j a(\phi_i, \phi_j) = (f, \phi_i), \quad i = 1, \ldots, M,
\]

or in matrix form,

\[
A\xi = b,
\]

where \( \xi_j = \hat{u}(N_j), A_{ij} = a(\phi_i, \phi_j)_{\Omega} \) is SPD, and \( b_i = (f, \phi_i)_{\Omega} \).
Neumann Problem

- Let $\Omega$ be the unit square $(x, y) \in [0, 1] \times [0, 1]$
- Assume Neumann boundary conditions
- Then the 2D Possion problem is:
  Find $u \in V := H^1(\Omega)$ such that
  
  $$-\Delta u = f \quad \text{in } \Omega,$$
  $$\frac{\partial u}{\partial n} = g \quad \text{on } \partial \Omega.$$
Variational Formulation

Find \( u \in V \) such that

\[
a(u, v)_\Omega = (f, v)_\Omega + \langle g, v \rangle_{\partial \Omega}, \quad \forall v \in V,
\]

where

\[
a(u, v)_\Omega := \int_\Omega \nabla u \cdot \nabla v,
\]

\[
(f, v)_\Omega := \int_\Omega f v,
\]

\[
\langle g, v \rangle_{\partial \Omega} := \int_{\partial \Omega} g v.
\]

Note: used Green’s formula and \( \frac{\partial u}{\partial n} = g \) on \( \partial \Omega \).
$P_1$ Finite Element

We introduce a triangulation $T_h$ of $\Omega$ into triangles $K_i$, and a finite dimensional subspace:

$$V_h := \{ \hat{v} \in H^1(\Omega) : \hat{v}|_{K_i} \in P_1(K_i) \}.$$

Find $\hat{u} \in V_h$ such that

$$a(\hat{u}, \hat{v})_\Omega = (f, \hat{v})_\Omega + \langle g, \hat{v} \rangle_{\partial \Omega}, \quad \forall \hat{v} \in V_h.$$
Using Basis Functions

Representing \( \hat{u} \) and \( \hat{v} \) in terms of nodal basis functions \( \{ \phi_i \}_{i=0}^{M+1} \) of \( V_h \), we get the following system of algebraic equations:

\[
M+1 \sum_{\begin{align*}
j=0 \\
i=0, \ldots, M+1,
\end{align*}} \xi_j a(\phi_i, \phi_j) = (f, \phi_i) + \langle g, \phi_i \rangle_{\partial\Omega},
\]

or in matrix form,

\[
A\xi = b,
\]

where \( \xi_j = \hat{u}(N_j) \), \( A_{ij} = a(\phi_i, \phi_j)_{\Omega} \) is SPD, and

\[
b_i = (f, \phi_i)_{\Omega} + \langle g, \phi_i \rangle_{\partial\Omega}.
\]
Mixed Boundary

Let $\Omega$ be the unit cube, $\Gamma_0$ the face at $z = 0$, and $\Gamma_1 = \partial \Omega \setminus \Gamma_0$. Then we want to approximate $u$ in $\Omega$. 
Mixed Robin Boundary

Find \( u \in V \coloneqq H^1(\Omega) \) such that

\[
-\Delta u = f \quad \text{in } \Omega, \\
\alpha_0 u + \beta_0 \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_0, \\
\alpha_1 u + \beta_1 \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_1.
\]

Note: assume \( \beta_i \neq 0 \).
Variational Formulation

Find $u \in V$ such that

$$a(u, v)_\Omega - \left\langle \frac{\partial u}{\partial n}, v \right\rangle_{\Gamma_0} - \left\langle \frac{\partial u}{\partial n}, v \right\rangle_{\Gamma_1} = (f, v)_\Omega,$$

or

$$a(u, v)_\Omega + \alpha_0 \beta_0^{-1} \langle u, v \rangle_{\Gamma_0} + \alpha_1 \beta_1^{-1} \langle u, v \rangle_{\Gamma_1} = (f, v)_\Omega,$$

where $a(u, v)_\Omega$ and $(f, v)_\Omega$ are the same as above and

$$\langle u, v \rangle_{\Gamma_j} := \int_{\Gamma_j} uv, \quad j = 0, 1.$$
Finite Element Method

We introduce the triangulation $T_h$ as before, and the finite dimensional subspace

$$V_h := \{ \hat{u} \in H^1(\Omega) : \hat{u}|_{K_i} \in Q_1(K_i) \}$$

to get the finite element problem:
Find $\hat{u} \in V_h$ such that

$$a(\hat{u}, \hat{v})_\Omega + \alpha_0 \beta_0^{-1} \langle \hat{u}, \hat{v} \rangle_{\Gamma_0} + \alpha_1 \beta_1^{-1} \langle \hat{u}, \hat{v} \rangle_{\Gamma_1} = (f, \hat{v})_\Omega, \quad \forall \hat{v} \in V_h.$$
Using Basis Functions

Representing $\hat{u}$ and $\hat{v}$ in terms of nodal basis functions $\{\phi_i\}_{i=1}^M$ of $V_h$ we get the following matrix equation:

$$(A + G)\xi = b,$$

where $\xi$, $A$, and $b$ are similar to those above and

$$G_{ij}^0 := \langle \phi_i, \phi_j \rangle_{\Gamma_0},$$

$$G_{ij}^1 := \langle \phi_i, \phi_j \rangle_{\Gamma_1},$$

$$G := \frac{\alpha_0}{\beta_0} G^0 + \frac{\alpha_1}{\beta_1} G^1.$$
Parabolic and Elliptic

- Build off of elliptic FEM
- If boundary not moving, space-time rectangular in $t$ dimension
- Popular to use FEM for spatial discretization and FDM for time
- Performing FEM first results in semi-discrete formulation
- This is equivalent to a coupled system of ODEs
Scalar Wave Problem

Find \( u \in H^2([0, T]; L^2(\Omega)) \cap L^2([0, T]; H^1_0(\Omega)) \) such that

\[
\frac{1}{c^2}u_{tt} - \Delta u = f \quad \text{in } \Omega,
\]

\[
u = 0 \quad \text{in } \partial\Omega \times (0, T),
\]

\[
u(\cdot, 0) = u_0(\cdot) \quad \text{in } \Omega,
\]

\[
u_t(\cdot, 0) = u_1(\cdot) \quad \text{in } \Omega.
\]
Variational Formulation

Find \( u(\cdot,t) : [0, T] \rightarrow V := H_0^1(\Omega) \) such that

\[
\frac{1}{c^2}(u_{tt}, v)_\Omega + a(u, v)_\Omega = (f, v)_\Omega, \quad \forall v \in V,
\]

\[
(u(\cdot,0), v)_\Omega = (u_0(\cdot), v)_\Omega, \quad \forall v \in V,
\]

\[
(u_t(\cdot,0), v)_\Omega = (u_1(\cdot), v)_\Omega, \quad \forall v \in V.
\]
Semi-discrete Formulation

Find \( \hat{u}(\cdot, t) : [0, T] \rightarrow V_h \) such that

\[
\frac{1}{c^2}(\ddot{u}_h, \hat{v})_\Omega + a(\hat{u}, \hat{v})_\Omega = (f, \hat{v})_\Omega, \quad \forall \hat{v} \in V_h, \\
(\hat{u}(\cdot, 0), \hat{v})_\Omega = (u_0(\cdot), \hat{v})_\Omega, \quad \forall \hat{v} \in V_h, \\
(\hat{u}_h(\cdot, 0), \hat{v})_\Omega = (u_1(\cdot), \hat{v})_\Omega, \quad \forall \hat{v} \in V_h.
\]
Using Basis Functions

In matrix form,

\[
\frac{1}{c^2} \ddot{L}\xi(t) + A\xi(t) = b, \quad \forall t \in (0, T),
\]

\[
L\xi(0) = \chi^0,
\]

\[
L\dot{\xi}(0) = \chi^1,
\]

where \(\xi, A,\) and \(b\) are as above, and

\[
\chi_i^0 := (u_0, \phi_i)\Omega,
\]

\[
\chi_i^1 := (u_1, \phi_i)\Omega,
\]

\[
L_{ij} := (\phi_i, \phi_j)\Omega.
\]
Fully Discrete Formulation

In order to discretize in time, we introduce a (uniform) partition of the interval \([0,T]\):
\[
0 = t_0 < t_1 < \cdots < t_{NT} = T, \quad \text{and}
\]
\[
k := t_n - t_{n-1}, \quad n = 1, \ldots, NT.\]

\[
\frac{1}{c^2} L \frac{\xi^{n+1} - 2\xi^n + \xi^{n-1}}{k^2} + A\xi^n = b^n, \quad n = 1, \ldots, NT, \\
L\xi^0 = \chi^0, \\
L\xi^1 = \chi^0 + k\chi^1,
\]

where \(\xi^n_i \approx \xi_i(t_n)\) and \(b^n_i := b_i(t_n)\).
Mass Lumping

- Note that the fully discrete formulation is still implicit, thus a linear solve at each time step must be performed.

- Since $L_{ij} := (\phi_i, \phi_j)_{\Omega}$, it is possible to make $L$ diagonal by using a quadrature rule (Trapezoid) for the integration.

- The resulting explicit method is exactly the FDM.

- When to lump is an important question; numerical dispersion analysis can show, for example, in 1D consistent mass matrix requires a smaller time step than lumped, and a linear solve!

- Mass lumping can reduce accuracy, especially in higher dimensions.
Other Considerations

• Integration performed in “local” coordinates, then global matrix assembled.

• $K$ may be mapped to reference domain (e.g., $[-1, 1]$) for easy integration (especially quadrature rules).

• Technically speaking a finite element is a triple: geometric object, finite-dimensional linear function space, and a set of degrees of freedom.

• Many types of finite elements exist, including some with quadratic or cubic basis functions, first or second derivatives as degrees of freedom, or degrees of freedom in locations other than vertices (e.g., centroid).

• More general than FDM, and more easily applied to slanted or curved boundaries, especially involving normal derivative boundary conditions.
Conservation Laws

- Many PDEs are derived from physical models called *conservation laws*.

- The general principle is that the rate of change of $u(x, t)$ within a volume $V$ is equal to the flux past the boundary

$$\frac{\partial}{\partial t} \int_V u(x, t) + \int_{\partial V} f(u) \cdot n = 0$$

where $f$ is flux function.

- Nonlinear conservation laws can result in discontinuities in finite time even with smooth initial data.
Finite Volume Method

- Rather than pointwise approximations on a grid, FVM approximates the average integral value on a reference volume.

- Suppose region $V_i = [x_{i-(1/2)}, x_{i+(1/2)}]$ then

$$
\int_{x_{i-(1/2)}}^{x_{i+(1/2)}} u_t dx + f(u_{i+(1/2)}) - f(u_{i-(1/2)}) = 0
$$

where we have applied Gauss’s theorem and integrated analytically the resulting term $\int_{x_{i-(1/2)}}^{x_{i+(1/2)}} f_x(u) dx$.

- We can apply a quadrature rule, for example Midpoint, to the remaining integral to get a semi-discrete form

$$
(x_{i+(1/2)} - x_{i-(1/2)}) u_t(x_i) + f(u_{i+(1/2)}) - f(u_{i-(1/2)}) = 0.
$$
FVM Example

Consider the elliptic equation \( u_{xx} = f(x) \) on a control volume \( V_i = [x_{i-(1/2)}, x_{i+(1/2)}] \) then

\[
\int_{x_{i-(1/2)}}^{x_{i+(1/2)}} u_{xx} \, dx = \int_{x_{i-(1/2)}}^{x_{i+(1/2)}} f \, dx.
\]

Evaluating the left hand side analytically and the right via Midpoint gives

\[
u_x(x_{i+(1/2)}) - u_x(x_{i-(1/2)}) = (x_{i+(1/2)} - x_{i-(1/2)}) f_i
\]

Finally, using centered differences on the remaining derivatives yields

\[
\frac{u_{i+1} - 2u_i + u_{i-1}}{h} = hf_i
\]

for \( h = x_{i+(1/2)} - x_{i-(1/2)} \).
FVM Summary

- Applies to integral form of conservation law.
- Handles discontinuities in solutions.
- Natural choice for heterogeneous material as each grid cell can be assigned different material parameters.
- There exist theory for convergence, accuracy and stability.
Systems of Linear Equations

• For implicit methods must choose a linear solver.
• Direct (LU factorization)
  • More accurate
  • May be cheaper for many time steps
  • Banded (otherwise fill-in)
• Iterative
  • If accuracy less important than speed
  • Matrix-free
  • Sparse
  • SPD
Iterative Methods

- **Successive Over Relaxation (SOR)**
  - Simple to code
  - $\omega = 1$ is Gauss-Seidel

- **Conjugate Gradient**
  - SPD
  - Eigenvalues clustered together (Precondition)

- **Generalized Minimum Residual (GMRES)**
  - Non-SPD, e.g. convection-diffusion with upwinding
  - Krylov method: builds orthonormal basis which may get big (Restart)
  - Preconditioning helps (Incomplete Cholesky)