Homology classes that can be represented by embedded spheres in rational surfaces

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Abstract The problem of embedding spheres in rational surfaces $\mathbb{CP}^2 \# n\overline{\mathbb{CP}}^2$ is studied. For homology classes $u = (b_1 + k, b_2, \ldots, b_n)$ with positive self-intersection numbers, a necessary and sufficient condition to detect its representability is given when $k \leq 5$.

Keywords: four manifold, embedded sphere, self-intersection number.

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A well-known problem in topology of four manifolds is: given a two dimensional homology class $u \in H_2(M)$, can $u$ be represented by a smooth embedded sphere? When $M^{2n}$ is an $(n-1)$-connected manifold whose dimension is greater than 4, by Whitney trick, any $n$-homology class can be represented by a smooth embedded $S^n$. Due to the failure of Whitney trick in four dimensions, the problem becomes especially interesting. There are only few results before the 1980s (refer to refs. [1—5]). As the successful use of gauge theory in four dimensional topology, the situation changed drastically. For some “simple” manifolds like $\mathbb{CP}^2, n(S^2 \times S^2)(n \leq 3), n(\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2)(n \leq 3)$, $2\mathbb{CP}^2 \# 3\overline{\mathbb{CP}}^2$, complete results are obtained (refer to refs. [6—10]). For rational surfaces $\mathbb{CP}^2 \# n\overline{\mathbb{CP}}^2(n \leq 9)$, every nonnegative class (i.e. homology class with nonnegative self-intersection number) can be simplified to a reduced class by a given algorithm. The representability of a reduced class is known (refer to refs. [15—17]). For $n > 9$ partial result is obtained1) (refer refs. [11—17]). A very good exposition about this direction by Lawson can be found in ref. [18]. In this paper, we will give a simple condition to determine the representability of some classes in rational surfaces $\mathbb{CP}^2 \# n\overline{\mathbb{CP}}^2$ for any $n$. The virtues of our condition do not rely on the reduced class.

Denote by $O(1, n)$ the orthogonal group of Lorentz $(1, n)$ space, by $O(1, n)$ the integer subgroup of $O(1, n)$ (i.e. all entries in a matrix are integers). Let $\xi, \eta_1, \cdots, \eta_n$ be an orthonormal basis of Lorentz $(1, n)$ space, i.e. $\xi^2 = -\eta_k^2(i = 1 \cdots, n), \xi_k = \eta_i\eta_j = 0(i, j = 1, \cdots, n \text{ and } i \neq j)$. Let $S \in O(1, 2), R \in O(1, 3)$ be the following

---

1) After submitting this paper, we find that in theorem 4.2 of ref. [21] a standard form is given for arbitrary $n$ to those classes that can be represented by smooth embedded spheres.
transformations

\[
S:\begin{cases}
\xi \mapsto 3\xi + 2\eta_1 + 2\eta_2, \\
\eta_1 \mapsto -2\xi - \eta_1 - 2\eta_2, \\
\eta_2 \mapsto -2\xi - 2\eta_1 - \eta_2;
\end{cases}
\]

\[
R:\begin{cases}
\xi \mapsto 2\xi + \eta_1 + \eta_2 + \eta_3, \\
\eta_1 \mapsto -\xi - \eta_2 - \eta_3, \\
\eta_2 \mapsto -\xi - \eta_1 - \eta_3, \\
\eta_3 \mapsto -\xi - \eta_1 - \eta_2.
\end{cases}
\]

Then we have the following.

**Lemma 1**. \(\mathcal{O}(1, 2)\) is generated by \(S\) and trivial automorphisms. When \(3 \leq n \leq 9\), \(\mathcal{O}(1, n)\) is generated by \(R\) and trivial automorphisms, where trivial automorphisms are automorphisms changing the sign of \(\xi\) or \(\eta_i\) or interchanging \(\eta_i\) and \(\eta_j\).

Let \(u = a\xi - b_1\eta_1 \cdots - b_n\eta_n \in H_2(CP^2 \# nCP^2)\), where \(\xi, \eta_i\) are the canonical generators of \(CP^2\) and the \(i\)-th \(CP^2\). We let \(u\) be \(u = (a, b_1, \cdots, b_n) \in H_2(CP^2 \# nCP^2)\). Then we have \(\xi^2 = -\xi_i^2 = 1(i = 1, \cdots, n), \xi \eta_k = \eta_k \eta_j = 0(i, j = 1, \cdots, n\) and \(i \neq j\). Therefore \(H_2(CP^2 \# nCP^2)\) can be regarded as the Lorentz \((1, n)\) space over integers. Its isometry group is \(\mathcal{O}(1, n)\). We have:

**Lemma 2**. When \(n \leq 9\), each element in \(\mathcal{O}(1, n)\) is realized by a diffeomorphism of \(CP^2 \# nCP^2\).

From this Lemma we see that under the action of \(\mathcal{O}(1, n)\) all elements in an orbit have the same representability when \(n \leq 9\). In particular, \(S\) and \(R\) and trivial automorphisms can be realized by diffeomorphisms.

We now consider homology classes \(u = (a, b_1, \cdots, b_n) \in H_2(CP^2 \# nCP^2)\) with positive self-intersection number. As changing the sign of \(a, b_i\) does not change their representability, we may assume \(a, b_i\) nonnegative. Furthermore we can assume \(b_1 \geq b_2 \geq \cdots \geq b_n\). \(u^2 > 0\) implies \(u\) can be written as \(u = (b_1 + k, b_1, \cdots, b_n)\), where \(k = a - b_1 > 0\). In ref. [13] we have proved when \(k = 1, u\) is representable by a smooth embedded sphere. Here we consider the case \(1 < k \leq 5\) and have the following:

**Proposition 1**. Suppose \(u = (b_1 + 2, b_1, \cdots, b_n) \in H_2(CP^2 \# nCP^2)\) has positive self-intersection number and \(b_i \geq 0\). Then \(u\) is representable by a smooth embedded sphere iff

\[
b_1 = \frac{1}{4} \sum_{i=2}^{n} (b_i^2 - \alpha_i^2),
\]

where \(\alpha_i = 0\) or 1 is the modular 2 class of \(b_i\).

**Proposition 2**. Suppose \(u = (b_1 + 3, b_1, \cdots, b_n) \in H_2(CP^2 \# nCP^2)\) has positive self-intersection number and \(b_i \geq 0\). Then \(u\) is representable by smooth embedded
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sphere iff one of the following holds:

\[ b_1 = \frac{1}{6} \sum_{i=2}^{n} (b_i^2 - \alpha_i^2) - 1, \]

\[ b_1 = \frac{1}{6} \sum_{i=2}^{n} (b_i^2 - \alpha_i^2) - \frac{1}{2}, \]

where \( \alpha_i = -1, 0 \) or 1 and \( b_i \equiv \alpha_i \mod 3 \).

**Proposition 3.** Suppose \( u = (b_1 + 4, b_1, \cdots, b_n) \in H_2(CP^2 \# n\overline{CP}^2) \) has positive self-intersection number and \( b_i \geq 0 \). Then \( u \) is representable by smooth embedded sphere iff one of the following holds:

\[ b_1 = \frac{1}{8} \sum_{i=2}^{n} (b_i^2 - \alpha_i^2) - 1 \quad \text{and} \quad |\beta| = 3 \text{ or } 4, \]

\[ b_1 = \frac{1}{8} \sum_{i=2}^{n} (b_i^2 - \alpha_i^2) - 1 \quad \text{and} \quad |\beta| \leq 2, \#\{\alpha_i = 2, 2 \leq i \leq n\} = 1, \]

\[ b_1 = \frac{1}{8} \sum_{i=2}^{n} (b_i^2 - \alpha_i^2) \quad \text{and} \quad \#\{\alpha_i = 2, 2 \leq i \leq n\} = 3, \]

where \(-1 \leq \alpha_i \leq 2\), and \( b_i \equiv \alpha_i \mod 4 \); \(-3 \leq \beta \leq 4\), and \( b_2 - \sum_{i=2}^{n} (b_i - \alpha_i) \equiv \beta \mod 8 \).

**Proposition 4.** Suppose \( u = (b_1 + 5, b_1, \cdots, b_n) \in H_2(CP^2 \# n\overline{CP}^2) \) has positive self-intersection number and \( b_i \geq 0 \). Then \( u \) is representable by smooth embedded sphere iff one of the following holds:

\[ b_1 = \frac{1}{10} \sum_{i=2}^{n} (b_i^2 - \alpha_i^2) - 2, \]

\[ b_1 = \frac{1}{10} \sum_{i=2}^{n} (b_i^2 - \alpha_i^2) - \frac{3}{2}, \]

\[ b_1 = \frac{1}{10} \sum_{i=2}^{n} (b_i^2 - \alpha_i^2) - 1 \quad \text{and} \quad \#\{\alpha_i = 2, 2 \leq i \leq n\} = 3, \]

\[ b_1 = \frac{1}{10} \sum_{i=2}^{n} (b_i^2 - \alpha_i^2) - \frac{1}{2} \quad \text{and} \quad \#\{\alpha_i = 2, 2 \leq i \leq n\} = 4, \]

\[ b_1 = \frac{1}{10} \sum_{i=2}^{n} (b_i^2 - \alpha_i^2) \quad \text{and} \quad \#\{\alpha_i = 2, 2 \leq i \leq n\} = 6, \]

where \(|\alpha_i| \leq 2\), and \( b_i \equiv \alpha_i \mod 5 \).

The proof of propositions is based on the following Lemma.

**Lemma 3.** For \( m \geq 2 \), homology class \( u = (m + 2, m, 2, \alpha_3, \cdots, \alpha_n) \in H_2(CP^2 \# n\overline{CP}^2) \) with \( u^2 \geq 0 \) cannot be represented by an embedded sphere, where \(|\alpha_i| \leq 1\), \( i = 3, \cdots, n \).

**Proof.** \( CP^2 \# n\overline{CP}^2 \) has a structure of algebraic surfaces. There is a symplectic
form $\omega$ such that for any algebraic curve $c$ we have $\omega \cdot [c] > 0$. By the generalized adjunction formula\cite{20}, $2g(c) - 2 \geq K \cdot [c] + [c]^2$, where $K = (-3, -1, \cdots, -1)$ is the canonical class of $\mathbb{CP}^2\# n\mathbb{CP}^2$. Let $[c] = u$. Then $g(c) \geq 1$.

Similarly, we can show:

**Lemma 4.** For $m \geq 2$, homology class $u = (m + 3, m, \beta, \alpha_3, \cdots, \alpha_n) \in H_2(\mathbb{CP}^2\# n\mathbb{CP}^2)$ with $u^2 \geq 0$, cannot be represented by an embedded sphere, where $|\beta| \leq 3, |\alpha_i| \leq 1$.

**Proof of Proposition 1.** Without loss of generality, suppose $b_1 \geq \cdots \geq b_n \geq 0$. We divide the proof into three cases: $n = 2, n = 3$ and $n > 3$.

(I) $n = 2$. Note

$$(b_1 + k, b_1, b_2) \xrightarrow{S_1} (b_1 - 2b_2 + 3k, b_1 - 2b_2 + 2k, 2k - b_2) \xrightarrow{T_3} (b_1^{(1)} + k, b_1^{(1)}, b_2^{(1)}),$$

where $T_i$ is the automorphism of changing the sign of $(i + 1)$-th component.

Denote $S_1 = T_3S$ and $(b_1^{(i+1)} + k, b_1^{(i+1)}, b_2^{(i+1)}) = S_1((b_1^{(i)} + k, b_1^{(i)}, b_2^{(i)})$. Then $(b_1 + k, b_1, b_2) \xrightarrow{S^2_1} (b_1^{(n)} + k, b_1^{(n)}, b_2^{(n)})$, where

$$\begin{cases}
    b_1^{(n)} = b_1^{(n-1)} - 2b_2^{(n-1)} + 2k, \\
    b_2^{(n)} = b_2^{(n-1)} - 2k.
\end{cases}$$

Hence

$$\begin{cases}
    b_1^{(n)} = b_1 - 2nb_2 + 2kn^2, \\
    b_2^{(n)} = b_2 - 2kn.
\end{cases} \quad (1)$$

In the present situation, $k = 2$. We can choose suitable $n$ such that $-1 \leq b_2^{(n)} = b_2 - 2kn = \beta \leq 2$. Since the automorphism $S_1^n : u \mapsto u' = (b_1^{(n)} + k, b_1^{(n)}, b_2^{(n)})$ preserves Lorentz inner product, $u^2 = u'^2 = 4b_1^{(n)} - b_2^{(n)} + 4 \geq 0$. We see $b_1^{(n)} \geq 0$. Changing the sign of $b_2^{(n)}$ or the order of $b_1^{(n)}, b_2^{(n)}$ if required, we may regard $u'$ as a reduced class. By Lemma 2, all the transformations involved here can be realized by diffeomorphisms. By refs. [13,14,16], we see $u' = (b_1^{(n)} + 2, b_1^{(n)}, b_2^{(n)})$ is representable by an embedded sphere iff

$$\begin{cases}
    b_1^{(n)} = 0, \\
    b_2^{(n)} = 0, \pm 1;
\end{cases}$$

or

$$\begin{cases}
    b_1^{(n)} = 1, \\
    b_2^{(n)} = 2.
\end{cases}$$

Substituting it into (1), we get Proposition 1.

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(II) \( n = 3 \). Similar to the case \( n = 2 \). We use transformation \( R \) in Lemma 1 to get
\[
(b_1 + k, b_1, b_2, b_3) \xrightarrow{R} (b_1 - b_2 - b_3 + 2k, b_1 - b_2 + b_3 - k, b_3 - b_2 - k)
\]
where \( T_i \) is as before and \( T_{ij} \) is the transformation of changing the order of the \( i \)-th and the \( j \)-th components.

\[
\begin{align*}
  b_1^{(1)} &= b_1 - b_2 - b_3 + k, \\
  b_2^{(1)} &= b_2 - k, \\
  b_3^{(1)} &= b_3 - k.
\end{align*}
\]

Denote \( R_1 = T_3T_4T_34R, (b_1^{(n)} + k, b_1^{(n)}, b_2^{(n)}, b_3^{(n)}) \xrightarrow{R_1} (b_1^{(n+1)} + k, b_1^{(n+1)}, b_2^{(n+1)}, b_3^{(n+1)}) \). Then
\[
\begin{align*}
  b_1^{(n+1)} &= b_1^{(n)} - b_2^{(n)} - b_3^{(n)} + k, \\
  b_2^{(n+1)} &= b_2^{(n)} - k, \\
  b_3^{(n+1)} &= b_3^{(n)} - k.
\end{align*}
\]

Hence
\[
\begin{align*}
  b_1^{(n)} &= b_1 - n(b_2 + b_3) + n^2k, \\
  b_2^{(n)} &= b_2 - nk, \\
  b_3^{(n)} &= b_3 - nk. \\
\end{align*}
\]

In the present situation, \( k = 2 \). We can choose \( n \) such that \( b_3^{(n)} = b_3 - 2n = \alpha_3 \). Hence
\[
u = (b_1 + 2, b_1, b_2, b_3) \xrightarrow{R_1^n} (b_1^{(n)} + 2, b_1^{(n)}, b_2^{(n)}, b_3^{(n)}) = u_1.
\]
Choose \( m \) such that \(-1 \leq b_2^{(n)} - 4m = \beta \leq 2 \). Similar to the case of \( n = 2 \), use \( S''_1 \) to the first 3 components of \( u_1 \) while keeping the 4-th component fixed. This transformation is still denoted by \( S''_1 \). We have
\[
u_1 \xrightarrow{S''_1} \nu_2 = (b_1^{(m,n)} + 2, b_1^{(m,n)}, b_2^{(m,n)}, \alpha_3),
\]
where
\[
\begin{align*}
  b_1^{(m,n)} &= b_1^{(n)} - 2mb_2^{(n)} + 4m^2, \\
  b_2^{(m,n)} &= b_2^{(n)} - 4m = \beta.
\end{align*}
\]

Note \( b_3^{(n)} = b_3 - 2n = \alpha_3 \). By (2) and (3)
\[
b_1^{(m,n)} = b_1^{(n)} - \frac{b_2^{2} + b_3^{2} - \beta^2 - \alpha_3^2}{4}.
\]

From \( u^2 = u_2^2 = 4b_1^{(m,n)} + 4 - \beta^2 - \alpha_3^2 > 0 \) we deduce \( b_1^{(m,n)} \geq 0 \). Because \( R_1, S_1 \) preserve inner product, by refs. [13,14,16] and Lemma 3, \( u \) is representable iff
\[
\begin{align*}
  &b_1^{(m,n)} = 0, \\
  &|\beta| \leq 1, \\
  &\alpha \leq 1;
\end{align*}
\]

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or

\[
\begin{cases}
    b_{1}^{(m,n)} = 1, \\
    \beta = 2.
\end{cases}
\]

Both cases are equivalent to the result of Proposition 1.

(III) \( n > 3 \). Let \( b_{i} = 2m_{i} + \alpha_{i} \) \((i \geq 2)\). Similar to the case \( n = 3 \), use \( R_{1}^{m_{n}} \) to the first 4 components of \( u \) while keeping the rest components fixed. Denote the resulting class to be \( u_{1} = (b_{1}^{(1)} + k, b_{1}^{(2)} b_{2}^{(1)}, b_{3}^{(1)}, b_{4}, \cdots, b_{n}) \).

By (2), we have

\[
\begin{align*}
    b_{1}^{(1)} &= b_{1} - m_{3}(b_{2} + b_{3}) + m_{3}^{2}k, \\
    b_{2}^{(1)} &= b_{2} - m_{3}k, \\
    b_{3}^{(1)} &= b_{3} - m_{3}k = \alpha_{3},
\end{align*}
\]

where \( k = 2 \). Use \( R_{1}^{m_{n}} \) to the first, second, third and fifth components of \( u_{1} \) while keeping the rest components fixed. The resulting class is denoted to be \( u_{2} = (b_{1}^{(2)} + k, b_{1}^{(2)}, b_{2}^{(2)}, \alpha_{3}, b_{4}, \cdots, b_{n}) \). We have

\[
\begin{align*}
    b_{1}^{(2)} &= b_{1}^{(1)} - m_{4}(b_{1}^{(1)} + b_{4}) + m_{4}^{2}k, \\
    b_{2}^{(2)} &= b_{2}^{(1)} - m_{4}k, \\
    b_{3}^{(2)} &= b_{3} - m_{4}k = \alpha_{4}.
\end{align*}
\]

Similarly, \( u_{n-2} = (b_{1}^{(n-2)} + k, b_{1}^{(n-2)}, b_{2}^{(n-2)}, \alpha_{3}, \alpha_{4}, \cdots, \alpha_{n}) \), where

\[
\begin{align*}
    b_{1}^{(n-2)} &= b_{1}^{(n-3)} - m_{n}(b_{2}^{(n-3)} + b_{n}) + m_{n}^{2}k, \\
    b_{2}^{(n-2)} &= b_{2}^{(n-3)} - m_{n}k, \\
    b_{3}^{(n-2)} &= b_{3} - m_{n}k = \alpha_{n}.
\end{align*}
\]

Finally, use \( S_{1}^{m} \) to the first 3 components of \( u_{n-2} \) while keeping the rest components fixed, we obtain \( u_{n-1} = (b_{1}^{(n-1)} + k, b_{1}^{(n-1)}, b_{2}^{(n-1)}, \alpha_{3}, \alpha_{4}, \cdots, \alpha_{n}) \), where \( b_{2}^{(n-2)} = 2km + \beta, k = 2, \beta = 0, \pm 1 \) or 2. By (1),

\[
\begin{align*}
    b_{1}^{(n-1)} &= b_{1}^{(n-2)} - 2mb_{2}^{(n-2)} + 2km^{2}, \\
    b_{2}^{(n-1)} &= b_{2}^{(n-2)} - 2km = \beta.
\end{align*}
\]

By (6),

\[
b_{4}^{(n-2)} = b_{2} - \frac{n}{i=1} (b_{1} - \alpha_{i}).
\]

Eliminating \( m_{i}, m \) from (6), (7) and (8), we get

\[
b_{2}^{(n-1)} = b_{1} - \frac{1}{2k} \sum_{i=2}^{n} b_{i}^{2} + \frac{1}{2k} \left( \beta^{2} + \sum_{i=3}^{n} \alpha_{i}^{2} \right).
\]

Because all the above transformations preserve inner product and can be realized by diffeomorphisms. Hence \( u^{2} = u_{n-1}^{2} = 2kb_{1}^{(n-1)} + k^{2} - \beta^{2} - \sum_{i=3}^{n} \alpha_{i}^{2} > 0 \). When \( k = 2 \),
this deduces $b_1^{(n-1)} \geq 0$. By ref. [16] and Lemma 3, $u$ is representable by smooth spheres iff

$$
\begin{cases}
  b_1^{(n-1)} = 0, \\
  \beta = 0, \pm 1;
\end{cases}
$$

or

$$
\begin{cases}
  b_1^{(n-1)} = 1, \\
  \beta = 2.
\end{cases}
$$

From (7) and (8), by analyzing the relations between $\alpha_2$ and $\beta$ we see both cases are equivalent to the result of Proposition 1.

**Proof of Proposition 2.** By the same calculation as in Proposotion 1 we know (8) and (9) still hold. Now $k = 3$, $\beta$ satisfies

$$
\begin{cases}
  b_2^{(n-2)} - 2km = \beta, \\
  b_2^{(n-2)} = b_2 - \sum_{i=3}^{n} (b_i - \alpha_i),
\end{cases}
$$

where $|\beta| \leq 3$. Since $u^2 = u_{n-1}^2 \geq 0$, we see $b_1^{(n-1)} \geq -1$. By ref. [15] and Lemma 4, $u$ is representable by a smooth sphere iff

$$
\begin{cases}
  b_1^{(n-1)} = -1, \\
  \beta = 0, \pm 1;
\end{cases}
$$

or

$$
\begin{cases}
  b_1^{(n-1)} = 0, \\
  \beta = \pm 2;
\end{cases}
$$

or

$$
\begin{cases}
  b_1^{(n-1)} = 1, \\
  \beta = 3.
\end{cases}
$$

This is equivalent to the result of Proposition 2.

**Proof of Propositions 3 and 4.** Using similar transformation as in Proposition 1 we have $u' = (b_1^{(n-1)} + k, b_1^{(n-1)}, \beta, \alpha_3, \alpha_4, \cdots, \alpha_n)$ where $k = 4, \beta, \alpha_i$ as in this proposition. $b_1^{(n-1)}$, $\beta$ satisfy (7), (8) and (9). We can assume $\beta, \alpha_i$ nonnegative.

As all transformations involved preserve inner product and can be realized by diffeomorphisms, $u' = u'^2 = 8b_1^{(n-1)} + 16 - \beta^2 - \sum_{i=3}^{n} \alpha_i^2 > 0$. This deduces $b_1^{(n-1)} \geq -1$. So if $u$ is representable by a smooth sphere, by generalized adjuction formula

$$
b_1^{(n-1)} \leq \frac{1}{6} \left( \beta^2 - \beta + \sum_{i=3}^{n} (\alpha_i^2 - \alpha_i) \right) - 1.
$$

Let $r = \# \{ \alpha_i : |\alpha_i| = 2, i = 3, \cdots, n \}$. Then

$$
b_1^{(n-1)} \leq \frac{1}{6} \left( \beta^2 - \beta + \sum_{i=3}^{n} (\alpha_i^2 - \alpha_i) \right) - 1 = \frac{1}{6} (\beta^2 - \beta + 2r) - 1.
$$

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Hence $2r \geq 6(b_{1}^{(n-1)} + 1) - \beta^2 + \beta$. By $u^2 = 8b_{1}^{(n-1)} + 16 - \beta^2 - \sum_{i=3}^{n} \alpha_i^2 > 0$, we see $8b_{1}^{(n-1)} + 16 - \beta^2 > \sum_{i=3}^{n} \alpha_i^2 \geq 4r$. Therefore $b_{1}^{(n-1)} < \frac{1}{4} (\beta^2 - 2\beta + 4) \leq 3$. By generalized adjunction formula and refs. [13, 15], it is easy to verify that

- when $b_{1}^{(n-1)} = -1$, $u'$ is representable by a smooth sphere iff $\# \{\alpha_i : |\alpha_i| = 2, i = 3, \cdots, n\} = 1$;
- when $b_{1}^{(n-1)} = 0$, $u'$ is representable by a smooth sphere iff $|\beta| = 3$, or $|\beta| < 3$ and $\# \{\alpha_i : |\alpha_i| = 2, i = 2, \cdots, n\} = 3$;
- when $b_{1}^{(n-1)} = 1$, $u'$ is representable by a smooth sphere iff $|\beta| = 4$, or $|\beta| = 3$ and $\# \{\alpha_i : |\alpha_i| = 2, i = 2, \cdots, n\} = 3$;
- when $b_{1}^{(n-1)} = 2$, $u'$ is representable by a smooth sphere iff $|\beta| = 4$ and $\# \{\alpha_i : |\alpha_i| = 2, i = 2, \cdots, n\} = 3$.

Combining the above results, we get Proposition 3.

Similarly, when $k = 5$, by $u^2 > 0$ and generalized adjunction formula we have $-2 \leq b_{1}^{(n-1)} \leq 2$. Hence

- when $b_{1}^{(n-1)} = -2$, from $u^2 > 0$ we see $|\beta| \leq 2$. In this case $u$ is representable by a smooth sphere:
  - when $b_{1}^{(n-1)} = -1$, $|\beta| \leq 3$. $u$ is representable by a smooth sphere iff (1) $|\beta| = 3$, or (2) $|\beta| \leq 2$ and $\# \{\alpha_i : |\alpha_i| = 2, i = 2, \cdots, n\} = 3$;
  - when $b_{1}^{(n-1)} = 0$, $|\beta| \leq 4$. $u$ is representable by a smooth sphere iff (1) $|\beta| = 4$, or (2) $|\beta| = 3$ and $\# \{\alpha_i : |\alpha_i| = 2, i = 2, \cdots, n\} = 4$, or (3) $|\beta| = 2$ or 0 and $\# \{\alpha_i : |\alpha_i| = 2, i = 2, \cdots, n\} = 6$;
  - when $b_{1}^{(n-1)} = 1$, $|\beta| \leq 5$. $u$ is representable by a smooth sphere iff (1) $|\beta| = 5$, or (2) $|\beta| = 4$ and $\# \{\alpha_i : |\alpha_i| = 2, i = 2, \cdots, n\} = 4$;
  - when $b_{1}^{(n-1)} = 2$, $|\beta| \leq 5$. $u$ is representable by a smooth sphere iff $|\beta| = 5$ and $\# \{\alpha_i : |\alpha_i| = 2, i = 2, \cdots, n\} = 4$.

Combining the above results, we get Proposition 4.

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