Topics:

- Schwarz Inequality.
- Row and column vectors.
- Lines and planes.
- Systems of linear equations.
Cauchy-Bunyakovsky-Schwarz inequality
also known as Schwarz inequality:

$$|\vec{a} \cdot \vec{b}| \leq |\vec{a}||\vec{b}|$$

Easy to see in 2-D and 3-D spaces:

$$|\vec{a} \cdot \vec{b}| = |\vec{a}||\vec{b}| \cos \theta \leq |\vec{a}||\vec{b}|$$

since there $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta$.

Observe that it becomes equality if and only if $\theta = 0$ or $\pi$. 
In general, \(|\vec{a} \cdot \vec{b}| = |a_1 b_1 + a_2 b_2 + \cdots + a_n b_n| \leq |\vec{a}||\vec{b}|\) for \(\vec{a} = (a_1, a_2, \ldots, a_n)\) and \(\vec{b} = (b_1, b_2, \ldots, b_n)\)

**Example: 2-D** \(|\vec{a} \cdot \vec{b}| = |a_1 b_1 + a_2 b_2| \leq |\vec{a}||\vec{b}|\) as \((a_1 b_1 + a_2 b_2)^2 \leq (a_1^2 + a_2^2)(b_1^2 + b_2^2)\)

\[\uparrow\]

\[a_1^2 b_1^2 + 2a_1 b_1 a_2 b_2 + a_2^2 b_2^2 \leq a_1^2 b_1^2 + a_1^2 b_2^2 + a_2^2 b_1^2 + a_2^2 b_2^2\]

\[\uparrow\]

\[2a_1 b_1 a_2 b_2 \leq a_1^2 b_2^2 + a_2^2 b_1^2\]

\[\uparrow\]

\[0 \leq (a_1 b_2 - a_2 b_1)^2 = a_1^2 b_2^2 - 2a_1 b_1 a_2 b_2 + a_2^2 b_1^2\]
Since $|\vec{a} \cdot \vec{b}| = |a_1 b_1 + a_2 b_2 + \cdots + a_n b_n| \leq |\vec{a}| |\vec{b}|$ in $\mathbb{R}^n$,

$$-1 \leq \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} \leq 1$$

Consequently, there is a unique $\theta$ with $0 \leq \theta \leq \pi$ such that

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$$

in all dimensions.

We say that vectors $\vec{a}$ and $\vec{b}$ are orthogonal (perpendicular) if $\vec{a} \cdot \vec{b} = 0$

$$\vec{a} \perp \vec{b} \iff \vec{a} \cdot \vec{b} = 0$$
Row and column vectors.

Two ways to think of a vector:

row vectors: \( \vec{a} = (a_1, a_2, \ldots, a_n) \)

column vectors: \( \vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \)

Vectors are most often thought as column vectors in problems involving matrices.
• **Lines.**

A **line** \( L \) is determined by two points, \( P_0 \) and \( P \). Alternatively line \( L \) can be described by point \( P_0 \) and direction vector \( \vec{v} \).
• Parametric equation.

Given a line going through point $P_0$ in the direction $\vec{v}$ it can be described as the following **parametric equation**:

$$\vec{r} = \vec{r}_0 + t\vec{v}, \quad \text{for} \quad -\infty < t < \infty,$$

where $\vec{r}_0$ is the vector from the origin to $P_0$. 
• Parametric equation.

Parametric equation: \( \vec{r} = \vec{r}_0 + t\vec{v} \)

• Example: 2D. Given \( \vec{r}_0 = (x_0, y_0) \) and \( \vec{v} = (a, b) \), then

\[
(x, y) = (x_0, y_0) + t(a, b) = (x_0 + at, y_0 + bt)
\]
• **Example: 2D.** Given \( \vec{r}_0 = (x_0, y_0) \) and \( \vec{v} = (a, b) \), then
\[
(x, y) = (x_0, y_0) + t(a, b) = (x_0 + at, y_0 + bt)
\]
gives
\[
\begin{align*}
x &= x_0 + at \\
y &= y_0 + bt
\end{align*}
\]
for \(-\infty < t < \infty\).

These equations are called *(scalar) parametric equations* for the line \( L \).

The parametric equations for lines can be obtained similarly in 3-D, and higher dimensions. See the book.
Example: Find parametric equations for the line $L$ determined by the two points, $(2,5)$ and $(1,7)$.

Solution: Take $\vec{r}_0 = (2,5)$ (as $P_0 = (2,5)$ is a point on $L$), and

$$\vec{v} = (1 - 2, 7 - 5) = (-1, 2)$$

The parametric equation $\vec{r} = \vec{r}_0 + t\vec{v}$ reads

$$\begin{cases} x = 2 - t \\ y = 5 + 2t \end{cases}$$
• **Planes in 3-D.**

A plane $\Pi$ in 3-D space $\mathbb{R}^3$ is determined by a point $P_0 = (x_0, y_0, z_0)$ on it and a vector $\vec{n}$ perpendicular (normal) to $\Pi$. 
• **Planes in 3-D.**

Every point \((x, y, z)\) on the plane \(\Pi\) has to satisfy

\[
(x - x_0, y - y_0, z - z_0) \perp \vec{n}
\]

In other words, \((x - x_0, y - y_0, z - z_0) \cdot \vec{n} = 0\), which we write as

\[
a(x - x_0) + b(y - y_0) + c(z - z_0) = 0
\]
Equation \[ a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \]

• Example: Find an equation for the plane in 3-D that contains point \((1, 0, 2)\) and is perpendicular to vector \(\vec{n} = (4, 2, -1)\).

Answer: We use \((x - x_0, y - y_0, z - z_0) \cdot \vec{n} = 0\) with \((x_0, y_0, z_0) = (1, 0, 2)\), obtaining

\[
4(x - 1) + 2(y - 0) - (z - 2) = 0
\]

which we simplify to

\[
4x + 2y - z - 2 = 0
\]