Invariant Galton-Watson trees

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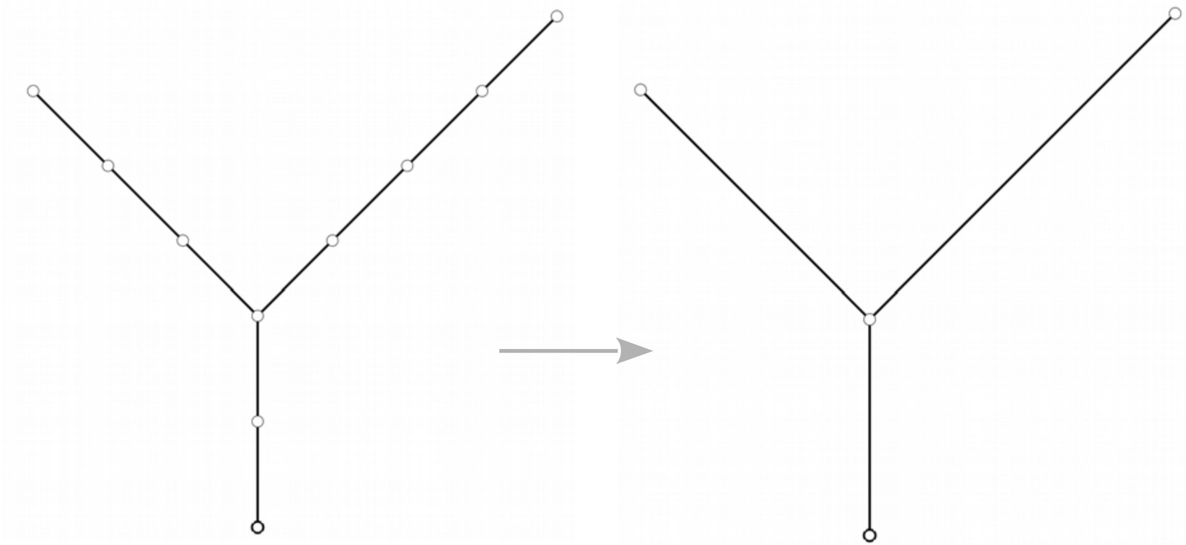
Combinatorial trees.

$\mathcal{T}$ - space of finite unlabeled rooted reduced trees.

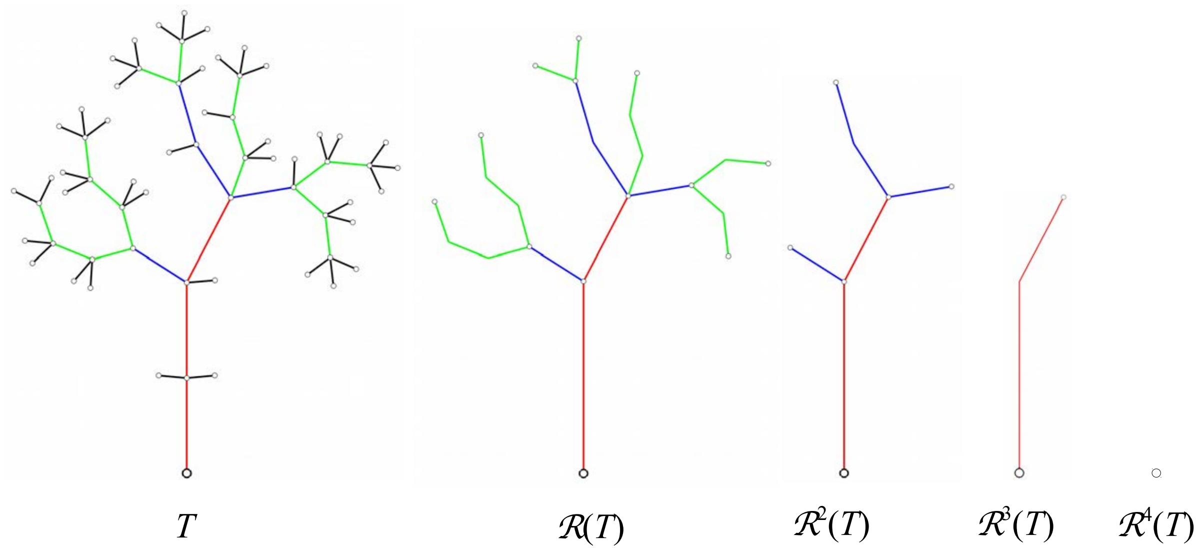
Empty tree $\phi = \{\rho\}$ comprised of a root vertex ρ and no edges.

$\mathcal{T}^{|}$ - subspace of $\mathcal{T}$ containing ϕ and all the trees in $\mathcal{T}$ with a stem (ρ has exactly one offspring).



(a) Original tree (b) Tree after series reduction

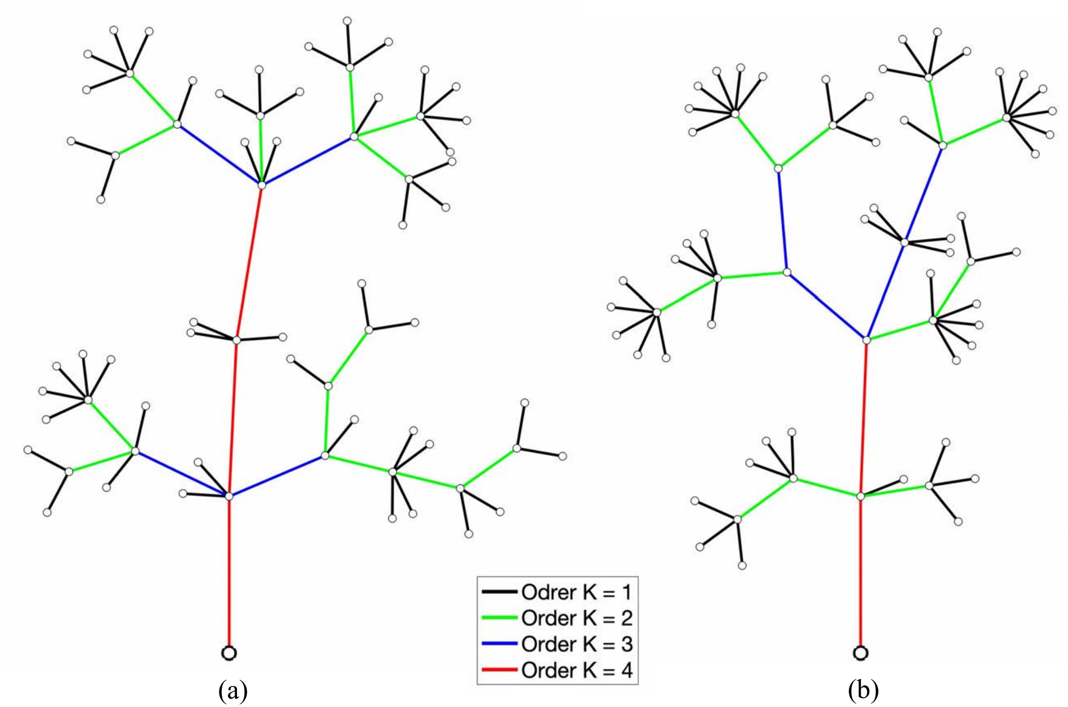
Horton pruning and Horton-Strahler order



Horton pruning $\mathcal{R} : \mathcal{T}^| \to \mathcal{T}^|$ is an onto function whose value $\mathcal{R}(T)$ for a tree $T \neq \phi$ is obtained by removing the leaves and their parental edges from T, followed by series reduction. We also set $\mathcal{R}(\phi) = \phi$.

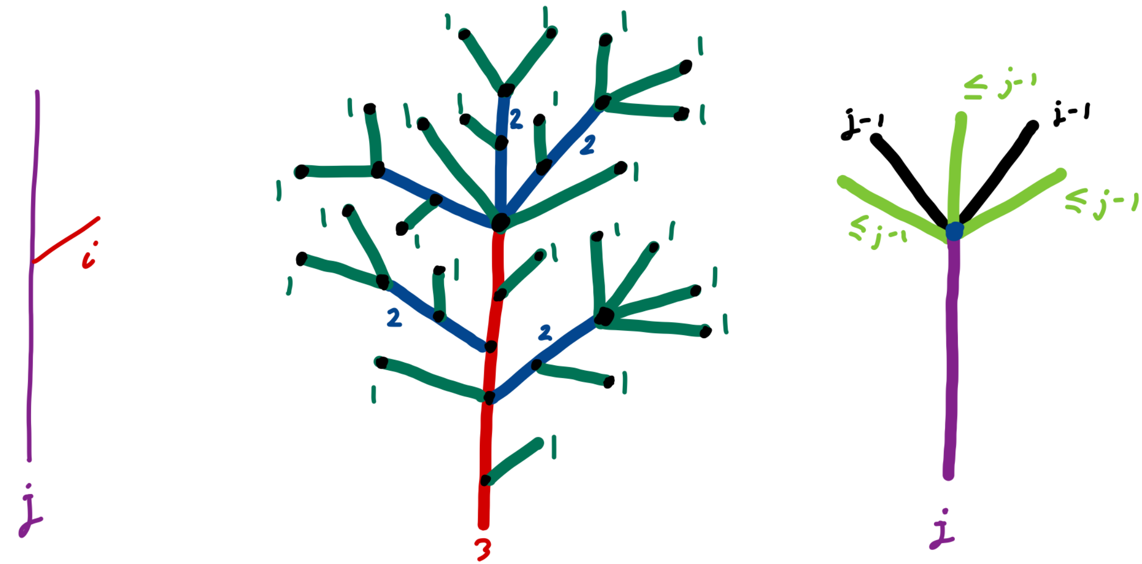
Horton-Strahler order: $\text{ord}(T) = \min\left\{k \geq 0 \ : \ \mathcal{R}^k(T) = \phi\right\}$.

Horton-Strahler order



Horton-Strahler order: $\text{ord}(T) = \min\left\{k \geq 0 \ : \ \mathcal{R}^k(T) = \phi\right\}$.

Side-branching.



Horton Law : $\dfrac{\mathcal{N}_{j-1}}{\mathcal{N}_j} \approx R$ Tokunaga self-similarity : $T_{i,j} = \dfrac{\mathcal{N}_{i,j}}{\mathcal{N}_j} \approx a\, c^{j-i-1}$

Horton laws. Side-branching. Tokunaga indices.

For $T \in \mathcal{T}^|$, let

$$N_i[T] = \text{number of order } i \text{ branches in } T$$

Let $\mathcal{N}_j[K] = E_K\Big[N_j[T]\Big]$ be the expected number of order j branches in a random tree T conditioned on $\text{ord}(T) = K$.

Horton law: there exists Horton exponent R such that

$$\lim_{K \to \infty} \frac{\mathcal{N}_j[K]}{\mathcal{N}_1[K]} = R^{1-j}$$

For $i < j$, let

$N_{i,j}[T] =$ number of order i side-branches of order j branches in T

Let $\mathcal{N}_{i,j}[K] = E_K\Big[N_{i,j}[T]\Big]$ be the expected number of order i side-branches of order j branches in T conditioned on $\text{ord}(T) = K$.

Tokunaga self-similarity: there exists Tokunaga indices $a > 0$ and $c > 0$ such that

$$T_{i,j}[K] = \frac{\mathcal{N}_{i,j}[K]}{\mathcal{N}_j[K]} = T_{j-i}, \text{ where } T_k = a\, c^{k-1}.$$

Reference: YK and I. Zaliapin, *Probability Surveys* (2020)

Critical binary Galton-Watson tree.

Let T be a **critical binary Galton-Watson tree**: $T \overset{d}{\sim} \mathcal{GW}(q_0 = q_2 = 1/2)$

• P. Flajolet, J.-C. Raoult, and J. Vuillemin, *TCS* (1979)

$$\mathsf{E}\left[\mathrm{ord}(T) \,\middle|\, N_1[T] = n\right] = \log_4 n + D\Big(\log_4 n\Big) + o(1), \quad \text{as} \quad n \to \infty,$$

where $D(\cdot)$ is a particular explicitly derived continuous periodic function of period one. This is a precursor of Horton law with $R = 4$:

$$\mathcal{N}_1[K] \asymp R^K \quad \Leftrightarrow \quad \lim_{K\to\infty} \frac{\mathcal{N}_j[K]}{\mathcal{N}_1[K]} = \lim_{K\to\infty} \frac{\mathcal{N}_1[K-j+1]}{\mathcal{N}_1[K]} = R^{1-j}$$

• G. A. Burd, E. C. Waymire, R. D. Winn, *Bernoulli* (2000)
Tokunaga sequence $T_k = 2^{k-1}$, i.e., Tokunaga self-similarity holds with $(a,c) = (1,2)$. Horton law $\lim_{K\to\infty} \frac{\mathcal{N}_j[K]}{\mathcal{N}_1[K]} = R^{1-j}$ holds with exponent $R = 4$.

Moreover, the following strong Horton law holds: for any $\epsilon > 0$

$$P\left(\left|\frac{N_j[T]}{N_1[T]} - R^{1-j}\right| > \epsilon \,\middle|\, \mathrm{ord}(T) = K\right) \to 0 \quad \text{as} \quad K \to \infty.$$

Horton prune-invariance

Consider a measure μ on $\mathcal{T}$ (or $\mathcal{T}^{|}$) such that $\mu(\phi) = 0$. Let ν be the pushforward measure, $\nu = \mathcal{R}_*(\mu)$, i.e.,

$$\nu(T) = \mu \circ \mathcal{R}^{-1}(T) = \mu\Big(\mathcal{R}^{-1}(T)\Big).$$

Measure μ is said to be Horton prune-invariant if for any tree $T \in \mathcal{T}$ (or $\mathcal{T}^{|}$) we have

$$\nu\,(T\,|T \neq \phi) = \mu(T).$$

Objective: finding and classifying Horton prune-invariant tree measures.

Attractors

For a tree measure ρ_0 let $\nu_k = \mathcal{R}_*^k(\rho_0)$ denote the pushforward probability measure induced by operator $\mathcal{R}^k$, i.e.,

$$\nu_k(T) = \rho_0 \circ \mathcal{R}^{-k}(T) = \rho_0\Big(\mathcal{R}^{-k}(T)\Big), \text{ and set } \rho_k(T) = \nu_k\,(T\;|T \neq \phi).$$

If $\lim_{k\to\infty} \rho_k(T) = \rho^*(T)\;\; \forall T \in \mathcal{T}$, then measure ρ^* is an attractor.

Objective: finding and classifying attractors.

Pruning Galton-Watson trees

Consider a Galton-Watson measure $\mathcal{GW}(\{q_k\})$ with $q_1 = 0$.

Assume criticality or subcriticality, i.e., $\sum_{k=0}^{\infty} kq_k \leq 1$.

Theorem. [G. A. Burd, E. C. Waymire, R. D. Winn, *Bernoulli* (2000)]

• Assume finite second moment, i.e., $\sum_{k=0}^{\infty} k^2 q_k < \infty$.

Galton-Watson measure $\mathcal{GW}(\{q_k\})$ is Horton prune-invariant if and only if it is critical binary Galton-Watson $\mathcal{GW}(q_0 = q_2 = 1/2)$.

• Assume criticality and finite branching, i.e., $\Big|\{k : q_k > 0\}\Big| < \infty$.
Let $\rho_0 \equiv \mathcal{GW}(\{q_k\})$, $\nu_k = \mathcal{R}_*^k(\rho_0)$, and set $\rho_k(T) = \nu_k\left(T \,|T \neq \phi\right)$.
Then,

$$\lim_{k\to\infty} \rho_k(T) = \rho^*(T) \qquad \forall T \in \mathcal{T},$$

where $\rho^* = \mathcal{GW}(q_0 = q_2 = 1/2)$ is critical binary Galton-Watson measure.

• If $\rho_0 \equiv \mathcal{GW}(\{q_k\})$ is subcritical, then $\rho_k(T)$ converges to a point mass measure, $\rho^* = \mathcal{GW}(q_0 = 1)$.

Pruning Galton-Watson trees

Consider a Galton-Watson measure $\mathcal{GW}(\{q_k\})$ with $q_1 = 0$.
Assume criticality or subcriticality, i.e., $\sum_{k=0}^{\infty} k q_k \leq 1$.

Let $Q(z) = \sum_{m=0}^{\infty} z^m q_m$ denote the generating function.

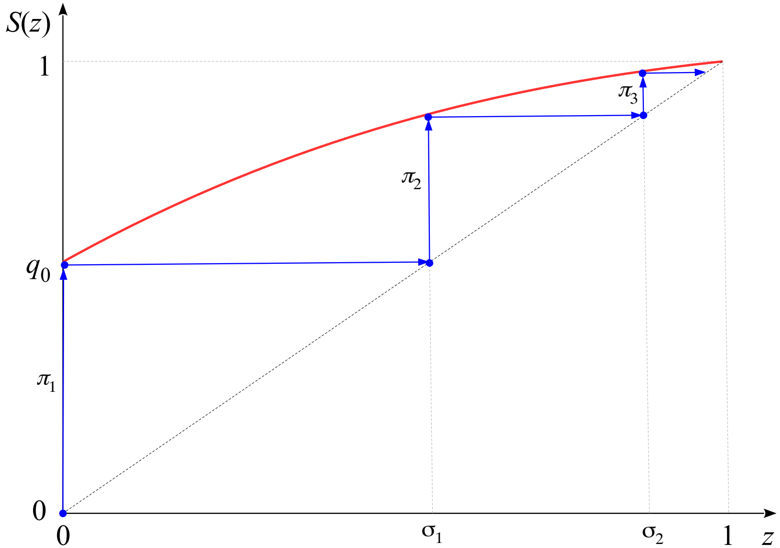
For $T \stackrel{d}{\sim} \mathcal{GW}(\{q_k\})$ let $\pi_j := P\big(\text{ord}(T) = j\big)$,

$$\sigma_0 = 0 \quad \text{and} \quad \sigma_j := \sum_{i=1}^{j} \pi_i \quad \forall j \geq 1$$

Lemma. [YK and I. Zaliapin, *Bernoulli* (2021)]

$$\sigma_j = \underbrace{S \circ \ldots \circ S}_{j \text{ times}}(0), \qquad \text{where} \qquad S(z) = \frac{Q(z) - zQ'(z)}{1 - Q'(z)}.$$

Pruning Galton-Watson trees



For $T \overset{d}{\sim} \mathcal{GW}(\{q_k\})$, let $\pi_j := P\Big(\text{ord}(T) = j\Big)$ and $\sigma_j := \sum_{i=1}^{j} \pi_i$

Lemma. [YK and I. Zaliapin, *Bernoulli* (2021)]

$$\sigma_j = \underbrace{S \circ \ldots \circ S}_{j \text{ times}}(0), \qquad \text{where} \qquad S(z) = \frac{Q(z) - zQ'(z)}{1 - Q'(z)}.$$

Regularity condition

Many of the results are proven under the following assumption.

Assumption 1. The following limit exists:

$$S'(1) = \lim_{x \to 1-} \frac{1 - S(x)}{1 - x} \quad \Leftrightarrow \quad \lim_{x \to 1-} \frac{Q(x) - x}{(1 - x)\big(1 - Q'(x)\big)} = 1 - S'(1)$$

Proposition. [YK and I. Zaliapin, *Bernoulli* (2021)]
If $\mathcal{GW}(\{q_k\})$ is a subcritical Galton-Watson measure with $q_1 = 0$, then Assumption 1 holds with $S'(1) = 0$.

Lemma. [YK and I. Zaliapin, *Bernoulli* (2021)]
Consider a critical Galton-Watson measure $\mathcal{GW}(\{q_k\})$ with $q_1 = 0$. If

$$\mathsf{E}[X^2] = \sum_{k=0}^{\infty} k^2 q_k < \infty \qquad \text{where} \quad X \stackrel{d}{\sim} \{q_k\},$$

then Assumption 1 holds with $S'(1) = \frac{1}{2}$.

Regularity condition

Lemma. [YK and I. Zaliapin, *Bernoulli* (2021)]
Consider a critical Galton-Watson measure $\mathcal{GW}(\{q_k\})$ with $q_1 = 0$ and infinite second moment, i.e., $\sum_{k=0}^{\infty} k^2 q_k = \infty$. Let $X \overset{d}{\sim} \{q_k\}$.

If the limit

$$\Lambda = \lim_{k\to\infty} \frac{k}{E[X \mid X \geq k]} = \lim_{k\to\infty} \frac{k \sum_{m=k}^{\infty} q_m}{\sum_{m=k}^{\infty} m q_m}$$

exists, then Assumption 1 holds with $S'(1) = \Lambda$.

Corollary. [YK and I. Zaliapin, *Bernoulli* (2021)]
Consider a critical Galton-Watson process $\mathcal{GW}(\{q_k\})$ with $q_1 = 0$ and offspring distribution $\{q_k\}$ of Zipf type:

$$q_k \sim Ck^{-(\alpha+1)} \quad \text{with} \quad \alpha \in (1,2] \quad \text{and} \quad C > 0.$$

Then Assumption 1 holds with $S'(1) = \Lambda = \frac{\alpha-1}{\alpha}$.

Invariant Galton-Watson measures

For a given $q \in [1/2, 1)$, a critical Galton-Watson measure $\mathcal{GW}(\{q_k\})$ is said to be the invariant Galton-Watson (IGW) measure with parameter q and denoted by $\mathcal{IGW}(q)$ if its generating function is given by

$$Q(z) = z + q(1 - z)^{1/q}.$$

Branching probabilities: $q_0 = q$, $q_1 = 0$, $q_2 = (1 - q)/2q$, and

$$q_k = \frac{1 - q}{k!\, q} \prod_{i=2}^{k-1} (i - 1/q) \quad (k \geq 3).$$

Here, if $q = 1/2$, then the distribution is critical binary, i.e., $\mathcal{GW}(q_0 = q_2 = 1/2)$.

If $q \in (1/2, 1)$, the distribution is of Zipf type with

$$q_k = \frac{(1 - q)\Gamma(k - 1/q)}{q\Gamma(2 - 1/q)\, k!} \sim C k^{-(1+q)/q}, \quad \text{where} \quad C = \frac{1 - q}{q\,\Gamma(2 - 1/q)}.$$

Invariant Galton-Watson measures

Recall

$$S(z) = \frac{Q(z) - zQ'(z)}{1 - Q'(z)}.$$

Assumption 1. The following limit exists:

$$S'(1) = \lim_{x \to 1-} \frac{1 - S(x)}{1 - x} \quad \Leftrightarrow \quad \lim_{x \to 1-} \frac{Q(x) - x}{(1 - x)\big(1 - Q'(x)\big)} = 1 - S'(1)$$

Horton prune-invariance: for $\nu(T) = \mu\Big(\mathcal{R}^{-1}(T)\Big)$,

$$\nu\,(T\,|T \neq \phi) = \mu(T).$$

Theorem. [YK and I. Zaliapin, *Bernoulli* (2021)] Consider a critical or subcritical Galton-Watson measure $\mathcal{GW}(\{q_k\})$ with $q_1 = 0$ that satisfies Assumption 1. Then, measure $\mathcal{GW}(\{q_k\})$ is Horton prune-invariant if and only if it is $\mathcal{IGW}(q_0)$.

Attraction property of critical Galton-Watson trees

Theorem. [YK and I. Zaliapin, *Bernoulli* (2021)]
Consider a critical Galton-Watson measure $\rho_0 \equiv \mathcal{GW}(\{q_k\})$ with $q_1 = 0$. Let $\nu_k = \mathcal{R}_*^k(\rho_0)$ denote the pushforward probability measure induced by operator $\mathcal{R}^k$, i.e.,

$$\nu_k(T) = \rho_0 \circ \mathcal{R}^{-k}(T) = \rho_0\Big(\mathcal{R}^{-k}(T)\Big), \quad \text{and set} \quad \rho_k(T) = \nu_k\left(T \mid T \neq \phi\right).$$

Suppose Assumption 1 is satisfied. Then,

$$\lim_{k\to\infty} \rho_k(T) = \rho^*(T),$$

where ρ^* denotes $\mathcal{IGW}(q)$ with $q = 1 - S'(1)$.

Finally, if $\rho_0 \equiv \mathcal{GW}(\{q_k\})$ is subcritical, then $\rho_k(T)$ converges to a point mass measure, $\mathcal{GW}(q_0 = 1)$.

Attraction property of critical Galton-Watson trees

Corollary. [YK and I. Zaliapin, *Bernoulli* (2021)]
Consider a critical Galton-Watson measure $\rho_0 \equiv \mathcal{GW}(\{q_k\})$ with $q_1 = 0$, with offspring distribution q_k of Zipf type:

$$q_k \sim Ck^{-(\alpha+1)} \quad \text{with} \quad \alpha \in (1,2] \quad \text{and} \quad C > 0.$$

Let $\nu_k = \mathcal{R}_*^k(\rho_0)$ and $\rho_k(T) = \nu_k\,(T\;|T \neq \phi)$.

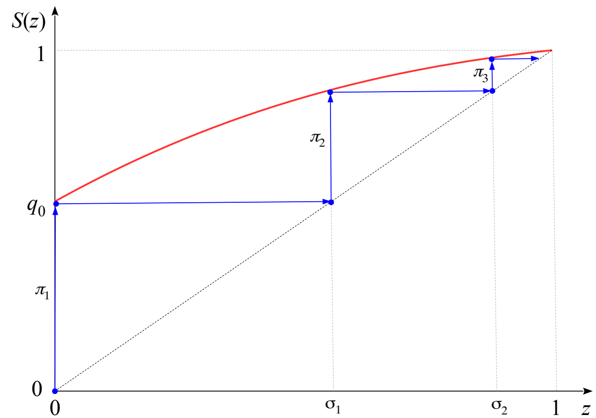
Then, $\lim_{k\to\infty} \rho_k(T) = \rho^*(T)$, where ρ^* is $\mathcal{IGW}(q)$ with $q = \frac{1}{\alpha}$.

Corollary. [YK and I. Zaliapin, *Bernoulli* (2021)]
Consider a critical Galton-Watson measure $\rho_0 \equiv \mathcal{GW}(\{q_k\})$ with $q_1 = 0$ such that $\sum_{k=2}^{\infty} k^2 q_k \;< \infty$.

Let $\nu_k = \mathcal{R}_*^k(\rho_0)$ and $\rho_k(T) = \nu_k\,(T\;|T \neq \phi)$.

Then, $\lim_{k\to\infty} \rho_k(T) = \rho^*(T)$, where ρ^* is $\mathcal{IGW}(1/2)$ (critical binary).

A gift from anonymous referee



For $\mathcal{IGW}(q_0)$, $S(z) = q_0 + (1 - q_0)z$

This is an example of Horton prune-invariant critical Galton-Watson measure $\mathcal{GW}(\{q_k\})$ for which Assumption 1 does not hold.

Let $q_0 \in (1/2, 1)$, $q_1 = 0$, and

$$q_m = \frac{1}{m!A} \sum_{n \in \mathbb{Z}} B^n \rho^{nm} e^{-\rho^n} \qquad m = 2, 3, \ldots,$$

where $\rho = 1 - q_0$, parameter $B \in \left((1-q_0)^{-1}, (1-q_0)^{-2}\right)$ is found by solving

$$\sum_{n \in \mathbb{Z}} B^n \left(1 - \rho^{n+1} - (1 + \rho^n - \rho^{n+1})e^{-\rho^n}\right) = 0, \quad \text{and} \quad A = \sum_{n \in \mathbb{Z}} B^n \rho^n \left(1 - e^{-\rho^n}\right).$$

Tokunaga coefficients and Horton law.

Lemma. [YK and I. Zaliapin, *Bernoulli* (2021)]
For a given $q \in [1/2, 1)$, consider an invariant Galton-Watson measure $\mathcal{IGW}(q)$. Then, its Tokunaga coefficients are

$$T^o_{i,j} = \frac{\mathcal{N}^o_{i,j}[K]}{\mathcal{N}_j[K]} = T^o_{j-i}, \quad \text{where} \quad T^o_k = c^{k-1} \ (k \geq 1) \quad \text{with} \quad c = \frac{1}{1-q}.$$

Additionally, $\pi_i = P\Big(\mathrm{ord}(T) = j\Big) = q\, c^{1-i}$, and the strong Horton law $\lim\limits_{K\to\infty} \frac{\mathcal{N}_k[K]}{\mathcal{N}_1[K]} = R^{1-k}$ holds with Horton exponent

$$R = c^{c/(c-1)} = (1-q)^{-1/q}.$$

Critical binary: since $\mathcal{IGW}(1/2) = \mathcal{GW}\Big(q_0 = q_2 = 1/2\Big)$, for $q = 1/2$, we have

$$c = 2, \quad \pi_i = 2^{-i}, \quad T_k = 2^{k-1}, \quad \text{and} \quad R = 4.$$