Critical percolation and Lorentz lattice gas model: an expository talk

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**Percolation** For each edge of the $d$-dimensional square lattice $\mathbb{Z}^d$ in turn, we declare the edge *open* with probability $p$ and *closed* with probability $1 - p$, independently of all other edges.
If we delete the closed edges, we are left with a random subgraph of $\mathbb{Z}^d$. A connected component of the subgraph is called a "cluster", and the number of edges in a cluster is the "size" of the cluster.
\[ \theta(p) \equiv P_p[0 \leftrightarrow \infty] \]

Obviously \( \theta(0) = 0 \) and \( \theta(1) = 1 \).

\( \exists \) critical \( 0 < p_c < 1 \) such that

- \( \theta(p) = 0 \) if \( p < p_c \) \( \Leftrightarrow \) subcritical model and
- \( \theta(p) > 0 \) if \( p > p_c \) \( \Leftrightarrow \) supercritical model

Standard reference:
Increasing events.

Configurations: \( \omega = \{ \omega(e) : e \in \mathbb{E}^d \} \), where

\[ \omega(e) = 1 \iff e \text{ is open}; \]

\[ \omega(e) = 0 \iff e \text{ is closed}. \]

Sample space: \( \Omega = \{0, 1\}^{\mathbb{E}^d} \).

**Partial Order:** we say \( \omega_1 \leq \omega_2 \) if and only if \( \omega_1(e) \leq \omega_2(e) \) for all \( e \in \mathbb{E}^d \).

**Def.** A random variable \( X \) is increasing if

\[ X(\omega_1) \leq X(\omega_2) \text{ whenever } \omega_1 \leq \omega_2 \]

**Def.** An event \( A \) is increasing if its indicator variable \( 1_A \), given by

\[ 1_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases} \]

is increasing.
**Russo's formula.** Given a configuration $\omega \in \Omega$ and an event $A$, an edge $e$ is **pivotal** if changing

$$\omega(e) \rightarrow 1 - \omega(e)$$

will determine whether the configuration is in $A$ or $A^C$; i.e. either

$$\omega \in A, \quad \{\text{changed } \omega\} \in A^C$$

or

$$\omega \in A^C, \quad \{\text{changed } \omega\} \in A.$$

Suppose $A$ depends on **finitely** many edges, and let $N_A := \# \text{ of pivotal edges}.$

**Thm.** (Russo's formula) If $A$ is increasing, then

$$\frac{d}{dp} P_p(A) = E_p[N_A]$$
**Exponential decay:** M.V. Menshikov (1986), enhanced - M. Aizenman and D.J. Barsky (1987)

**Thm.** If $p < p_c$ then $\exists \psi(p) > 0$ such that

$$P_p[0 \leftrightarrow \partial B(n)] < e^{-n\psi(p)} \text{ for all } n.$$ 

Proof outline: let $K_n = \{0 \leftrightarrow \partial B(n)\}$, then by Russo’s formula

$$\frac{d}{dp} P_p(K_n) = E_p[N_{K_n}] = \frac{1}{p} E_p[N_{K_n}|K_n] P_p(K_n).$$

Integrating $\frac{d}{dp} P_p(K_n)/P_p(K_n)$, get

$$P_a(K_n) = P_b(K_n)e^{-\int_a^b \frac{1}{p} E_p[N_{K_n}|K_n]dp},$$

where $0 \leq a < b \leq 1$. It can be showed that $E_p[N_{K_n}|K_n]$ growth roughly linearly in $n$ when $p < p_c$. 

2D Lorentz Lattice Gas (LLG) model. We place two-sided mirrors on the vertices of \( \mathbb{Z}^2 \) according to the following law: for \( 0 \leq p \leq 1 \), place

a NW mirror or NE mirror with probability \( \frac{p}{2} \) each.

Place NO MIRROR with probability \( 1 - p \) with probability \( = 1 - p \).
\[ \eta(p) = P_p \text{(the light ray returns to origin)}. \]

\[ \eta(0) = 0 \]

Grimmett: \( \eta(1) = 1 \), using \( p_c = \frac{1}{2} \) for 2D bond percolation model.
Russo’s formula adapted for LLG model.

Here $A_n = \{ \text{light cycle reaches } \partial B(n) \}$ is not increasing.

Need: a substitute property for “increasing”.

Consider $V = V(n)$ - the set of vertices inside the box $B(n)$.

Let $\Omega_V \equiv \{-1, 0, 1\}^V$ be the states space:

“$-1$” corresponds to NW mirror,

“1” to NE mirror

“0” to placing no mirror at a vertex.
For a vertex $v \in V$, we let

$$
\omega^+_v(u) = \begin{cases}
\omega(u) & \text{if } u \neq v, \\
1 & \text{if } u = v;
\end{cases}
$$

$$
\omega^-_v(u) = \begin{cases}
\omega(u) & \text{if } u \neq v, \\
-1 & \text{if } u = v;
\end{cases}
$$

$$
\omega^0_v(u) = \begin{cases}
\omega(u) & \text{if } u \neq v, \\
0 & \text{if } u = v.
\end{cases}
$$
Types of “pivotal” vertices: For an event $E \subset \Omega_V$, we say that a vertex $v \in V$ is

- **pivotal** if
  \[
  \begin{cases}
  \omega_v^+ \in E, \\
  \omega_v^0 \notin E, \\
  \omega_v^- \in E.
  \end{cases}
  \]

- **pivotal$^+$** if
  \[
  \begin{cases}
  \omega_v^+ \in E, \\
  \omega_v^0 \in E, \\
  \omega_v^- \notin E;
  \end{cases}
  \]

- **pivotal$^-$** if
  \[
  \begin{cases}
  \omega_v^+ \notin E, \\
  \omega_v^0 \in E, \\
  \omega_v^- \in E.
  \end{cases}
  \]

and $v \in V$ is **indifferent** if either

\[
\begin{cases}
  \omega_v^+ \in E, \\
  \omega_v^0 \in E, \\
  \omega_v^- \in E.
\end{cases}
\quad \text{or} \quad
\begin{cases}
  \omega_v^+ \notin E, \\
  \omega_v^0 \notin E, \\
  \omega_v^- \notin E.
\end{cases}
\]
Important Observation: we notice that in case of the event $A_n$ there can be only pivotal, pivotal$^+$, pivotal$^-$ and indifferent vertices.

Thm. (K.) For $0 < p < 1$,

$$\frac{d}{dp} P_p(A_n) = \sum_{v \in V_n} P_p\{\{v \text{ pivotal}\}\} - \sum_{v \in V_n} P_p\{\{v \text{ pivotal}^+\}\}$$

that is

$$\frac{d}{dp} P_p(A_n) = \mathbb{E}_p[N(A_n)] - \mathbb{E}_p[N^+(A_n)].$$