THE LORENTZ LATTICE GAS MODEL

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ABSTRACT. We will use a combinatorial approach to analyze the path of a light beam in the two-dimensional lattice, $\mathbb{L}^2$, traveling through a randomly generated labyrinth of two-sided mirrors. We will look at this specific case of the Lorentz Lattice Gas Model (LLGM) and utilize an analogue of Russo's Formula to further explore some of the intriguing details of the recurrence phenomenon that arise in this model. We will simplify the question of recurrence by breaking it down into a two case scenario that is significantly more tangible, and provide a theorem that brings the recurrence hypothesis well within grasp.

1. INTRODUCTION

There is a very interesting question dating back to Hendrik Lorentz that deals with the behavior of a particle traveling through $\mathbb{R}^d$, encountering randomly placed reflecting obstacles along the way. One notorious version of this problem, concerning a particle traveling along the integer lattice, $\mathbb{L}^2$, has proven particularly difficult to solve. First of all, we begin with the lattice, $\mathbb{L}^2$, and randomly place two-sided, reflecting mirrors at each of the vertices in the lattice with a particular probability. A mirror can be placed at a vertex in one of two ways: a NW mirror can be applied that will deflect a northward bound particle westward, and respectively, a southward bound particle to the east, along the axes; similarly, a NE mirror will deflect a particle moving north to the east, and south to the west, etc. (see Figure 1). Obviously, if there is no mirror present, then the particle will continue through the vertex with its original trajectory, passing through un-deflected.

We will place a mirror at each individual vertex with a probability $0 \leq p \leq 1$, where a NW mirror and a NE mirror are equiprobable $\left(\text{i.e., } P(NW) = P(NE) = \frac{p}{2}\right)$. On the other side of things, the probability of having no mirror at a vertex is $(1-p)$, so we are guaranteed to have one of these three states for every vertex in the lattice. It is good to note at this point that the mirror assignments of vertices are completely independent from one another and are determined based upon the probability, p, alone. Using this method, we can randomly generate an infinite labyrinth of mirrors on $\mathbb{L}^2$ for any particular probability p. Now, let's consider what will happen if we shine a flashlight northward from the origin of our lattice and let its light beam run through the labyrinth, reflecting off each mirror it encounters. The question becomes: will this light beam ever return to the origin and, furthermore, will the path be periodic, that is visit only finitely many points? Also, if we observe that this phenomenon does occur, are we always guaranteed to have the light beam's path be periodic for any value of p?

Date: 14 August 2008.

This work was done during the Summer 2008 REU program in Mathematics at Oregon State University.

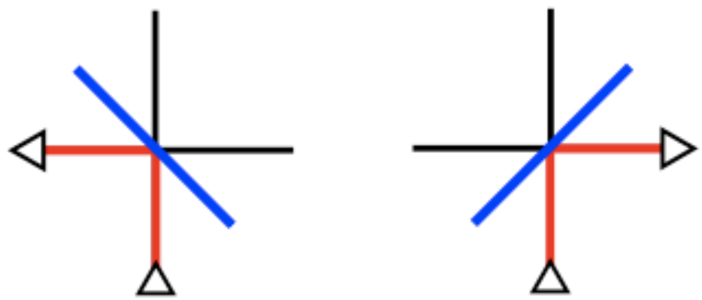

FIGURE 1. NW and NE Mirrors

While these questions seem relatively benign, the details of this system can be extremely difficult to describe mathematically. With that in mind, let's define exactly what we are looking for. We will let

$$\eta(p) = P_p(\text{ the light ray returns to the origin}), \tag{1}$$

and we wish to determine all values of p for which $\eta(p) = 1$. Obviously $\eta(0) = 0$, since this corresponds to the case where the lattice is completely devoid of mirrors and, subsequently, the light beam will continue northward indefinitely, unimpeded. It has also been well know for quite some time that $\eta(1) = 1$. Geoffrey Grimmett offers a very nice proof of recurrence for the $p = 1$ case in [2]. Unfortunately, those are the only two values of p for which the question of recurrence has been answered.

It has been conjectured by several mathematicians that $\eta(p) = 1$ for all $0 < p < 1$, but while some progress has been made using computational methods, it still remains a largely unsolved problem. Working toward a better understanding of this model for $0 < p < 1$ is the primary focus of this paper. As such, we will first provide the reader with an intuitive argument for why the recurrence hypothesis is almost certainly true, and then delve into a new mathematical approach that will simplify the question and introduce an aspect of the problem that will almost surely lead to a contradiction that will directly prove the recurrence hypothesis.

2. PRELIMINARY COMPUTATIONAL ANALYSIS

Before strictly analyzing the mathematics behind the Lorentz Lattice Gas Model problem, it is important to come to an understanding of what exactly is happening in our system, and hopefully get a better feeling for why so many mathematicians believe the recurrence hypothesis is true. The way in which we first analyze the problem is to look at finite regions of $\mathbb{L}^2$ to see what happens *on average* to light beams passing through random labyrinths generated for particular values of p. So, in order to talk further about some of this computational analysis, let's introduce a few definitions.

Definition 2.1. *Let D_n be the square region in the lattice with $n * n$ total vertices, where n is odd and our light origin is located at the center of the square, i.e. the vertex* $\left(\frac{(n-1)}{2}, \frac{(n-1)}{2}\right)$.

Now that we have a region, we need a way to describe what is happening inside of it.

Definition 2.2. *Let A_n be the event where a light beam, shone northward from the origin, reaches the boundary of our region D_n. Similarly, let A_n^c (read A_n complement) be the event where the light beam is completely contained in the region; that is to say it visits finitely many vertices and is periodic inside of the region.*

With these definitions in mind, we want to develop a way to computationally approximate

$$P_p(A_n^c) \ ,$$

the probability that for a given region, D_n, and a random labyrinth applied to it, a light beam from the origin will be completely contained in the region. We wrote a program in Matlab to help us out with the calculations. Essentially what the program does is randomly generate tens of thousands of labyrinths for a given probability p, then run light beams through each labyrinth, and find the proportion of light beams that stay inside of the region. This gives us a relatively accurate estimation of the probability of such an event occurring, and we can plot a curve to tell us what happens to the probability of the light beam being contained as we increase the size of the box.

Example 2.3. *Not surprisingly, for the case of p=1, the probability of A_n^c steadily approaches a value of* 1*. Using our program, we plotted the estimated probabilities for the values $3 \leq n \leq 401$, with a step size of* 2*, when $p = 1$ (Figure 2). Although these probabilities are obviously estimations, it is easy to see that this probability function will eventually approach* 1 *as $n \to \infty$, which is known to happen.*

The graph below is for relatively small values of n. To give the reader an idea of how slowly this function increases, we found that a box of size $n = 5003$ completely contained the light beam roughly 84.5% of the time for the $p = 1$ case. So, in order to get close to a probability of 1*, the box size must be extremely large.*

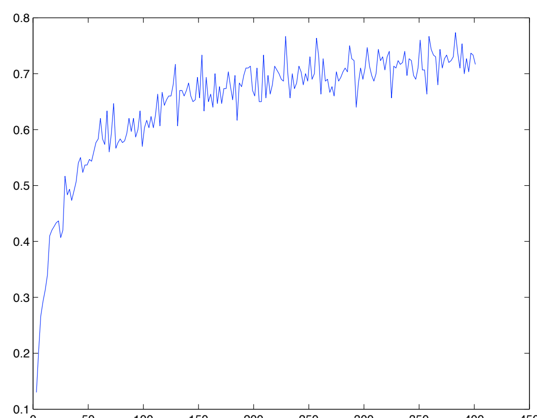

FIGURE 2. $P_p(A_n^c)$ as a function of n, for $p = 1$

Example 2.4. *Another example of what we have been working on computationally is the relationship between p values and the probability $P_p(A_n^c)$ for a given value of n. The graph (Figure*

3) below is the result of $1,000,000$ *trials on randomly generated labyrinths, with* $n = 5$*, for each value* $0 \leq p \leq 1$ *with a step size of* 0.05*. As the reader can see, it produces a very smooth curve. It makes sense that as we increase the value of p, and subsequently the number of mirrors in the labyrinth, the probability that the light beam is contained in the box must also increases.*

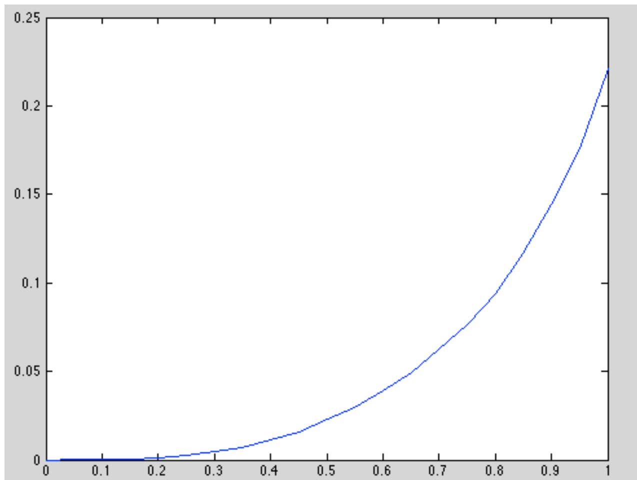

FIGURE 3. $P_p(A_5^c)$ as a function of p

From these analyses and others like them, we can get a sense that as the size of the box increases, the probability that the light beam will be entirely contained in it should increase proportionally. Similarly, we can witness the relationship that as we increase the number of mirrors in a given finite region, again the probability of the light beam being trapped inside the region will increase. So, intuitively, it makes sense that if we make our box extremely large and put a lot of mirrors in it (i.e., let p be very close to one), we should encounter a high probability of recurrence.

Now that the reader has somewhat of an intuitive understanding of what is happening in this model, let's look at some of the mathematical tools we have at our disposal that may help us to start chipping away at the problem.

3. NEW METHOD

Over the past several years, some progress has been made on this problem by Yevgeniy Kovchegov. In 2003 he wrote a paper concerning the recurrence phenomena of the LLGM [3] and he derived some useful results that directly pertain to the recurrence hypothesis. The remainder of this section will be dedicated to explaining some of his ideas and extending one of his theorems, to work toward a solution to the problem.

3.1. **An Introduction to Pivotal Points.** Let's start at the origin and again shine our flashlight northward. Consider that in order for that light beam to ever form a closed loop and be periodic (recurrent), it must return to *and travel northward from* the origin at another time step (actually infinitely many since it is periodic, but we are only interested in the first time it comes back). We can look at the mirror orientation of the origin to see what direction the end of the light beam must be traveling in order to close the loop off at the origin. If there is a NW mirror at the origin, the light beam must return to that vertex moving westward, so it will be reflected to the north; similarly, if it contains a NE mirror, the ray must be moving eastward; and if there is no mirror, obviously it has to be moving northward. So, what we will do now is shine two flashlights: one to the north and the other in the complementary direction we just figured out by looking at the mirror assignment at the origin. This will create two independent light paths that will travel through the labyrinth, undoubtedly cross, and, if the recurrence hypothesis holds, at some point meet each other in such a way as to form a closed loop.

Kovchegov focuses on a new class of points that are found by looking at the intersections of the two independent, yet complementary paths we just introduced. *Pivotal points* occur at vertices where the two independent and complementary paths from the origin intersect and either pass through each other (pivotal) or essentially bounce off one another due to a mirror being placed in between them (pivotal$^+$ or pivotal$^-$). This results in the two paths continuing to be independent rather than having their path closed off. There are always exactly two possible ways that the beams can avoid being closed off, and exactly one way to be closed off (NW, NE, or no mirror, depending on the situation) at any point of intersection of the two paths from the origin. [Note: it is advisable to take some time to convince oneself that this is true]

If a point of intersection of the two complementary beams contains a NW mirror (NE mirror), and subsequently the two paths are not closed, we call this a pivotal$^+$ (pivotal$^-$) point for the event that the two light rays reach the boundary of a given box. If at a point of intersection the two paths cross, as a result of no mirror being placed at the vertex, and continue on their respective ways, then we denote that vertex as pivotal for the event A_n. For an illustration of the differences between pivotal, pivotal$^+$, and pivotal$^-$ vertices, please direct your attention to Figure 4.

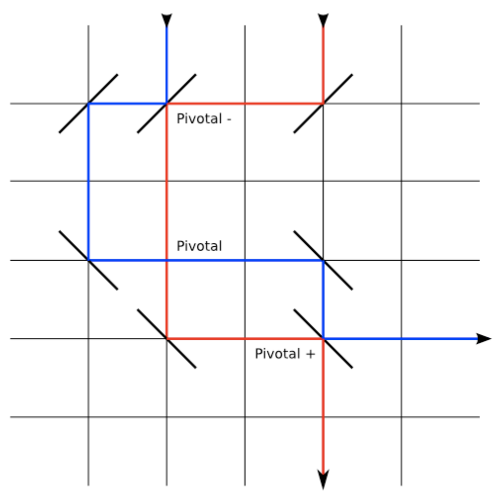

FIGURE 4. Three Types of Pivotal Points

In the following subsection we will introduce some new notation. We will let $\mathbb{E}_p[N(A_n)|A_n]$ denote the expected value of the number of pivotal points for two complementary paths from the origin, conditional on the event that both paths reach the boundary of the box (the event A_n).

Similarly, we will let $\mathbb{E}_p[N^+(A_n)|A_n])$ denote the expected value of pivotal$^+$ points, conditional on A_n. [Note: since NW and NE mirrors are equiprobable, $\mathbb{E}_p[N^+(A_n)|A_n]) = \mathbb{E}_p[N^-(A_n)|A_n])$]

3.2. **A Simplification of the LLGM Problem.** The following analysis was inspired by the arguments on pages 36-38 of [2]. Our goal here is to investigate *Theorem Two* from [3] and solve the differential equation with respect to A_n^c to see what we can find out about the structure of our labyrinths and the paths of the light beams passing through them. The theorem states:

$$\frac{d}{dp}P_p(A_n) = \left(\frac{1}{p}\mathbb{E}_p[N(A_n)|A_n] - \frac{2}{2-p}\mathbb{E}_p[N^+(A_n)|A_n]\right)P_p(A_n)\ . \tag{2}$$

Let's first take a look at $P_p(A_n^c)$ to see how we can relate it to the above theorem.

$$P_p(A_n^c) = 1 - P_p(A_n) \tag{3}$$

If we differentiate the above equation with respect to p we obtain the following result:

$$\frac{d}{dp}P_p(A_n^c) = -\frac{d}{dp}P_p(A_n).$$

So, if we make the appropriate substitutions, we have a restatement of Kovechegov's theorem:

Theorem 3.1. *For all n and $0 < p \leq 1$,*

$$\frac{d}{dp}P_p(A_n^c) = -\left(\frac{1}{p}\mathbb{E}_p[N(A_n)|A_n] - \frac{2}{2-p}\mathbb{E}_p[N^+(A_n)|A_n]\right)(1 - P_p(A_n^c))\ .$$

Now that we have the probabilities in terms of A_n^c we can go ahead and solve the differential equation for $P_p(A_n^c)$. So, first of all, let's separate the equation so we can integrate.

$$\frac{1}{(1-P_p(A_n^c))}\frac{d}{dp}P_p(A_n^c) = -\left(\frac{1}{p}\mathbb{E}_p[N(A_n)|A_n] - \frac{2}{2-p}\mathbb{E}_p[N^+(A_n)|A_n]\right)$$

$$\frac{1}{(1-P_p(A_n^c))}dP_p(A_n^c) = -\left(\frac{1}{p}\mathbb{E}_p[N(A_n)|A_n] - \frac{2}{2-p}\mathbb{E}_p[N^+(A_n)|A_n]\right)dp$$

Additionally, let's define our bounds of integration. We will let α and β be values of p such that $0 < \alpha < \beta \leq 1$. Thus, our integral becomes:

$$\int_\alpha^\beta \frac{1}{(1-P_p(A_n^c))}dP_p(A_n^c) = -\int_\alpha^\beta \left(\frac{1}{p}\mathbb{E}_p[N(A_n)|A_n] - \frac{2}{2-p}\mathbb{E}_p[N^+(A_n)|A_n]\right)dp\ .$$

Evaluating the integral on the left hand side from α to β we are left with

$$-\ln[1-P_\beta(A_n^c)] + \ln[1-P_\alpha(A_n^c)] = -\int_\alpha^\beta \left(\frac{1}{p}\mathbb{E}_p[N(A_n)|A_n] - \frac{2}{2-p}\mathbb{E}_p[N^+(A_n)|A_n]\right)dp$$

$$\ln[1-P_\alpha(A_n^c)] = \ln[1-P_\beta(A_n^c)] - \int_\alpha^\beta \left(\frac{1}{p}\mathbb{E}_p[N(A_n)|A_n] - \frac{2}{2-p}\mathbb{E}_p[N^+(A_n)|A_n]\right) dp\ .$$

Now, we can take the exponential of both sides, which leaves us with

$$[1-P_\alpha(A_n^c)] = \frac{[1-P_\beta(A_n^c)]}{exp\left(\int_\alpha^\beta \left(\frac{1}{p}\mathbb{E}_p[N(A_n)|A_n] - \frac{2}{2-p}\mathbb{E}_p[N^+(A_n)|A_n]\right) dp\right)}\ . \tag{4}$$

Additionally, let's plug $\beta = 1$ into the equation because we know a fair amount about the $p = 1$ case. For this reason, the substitution will undoubtedly simplify the equation.

$$[1-P_\alpha(A_n^c)] = \frac{[1-P_1(A_n^c)]}{exp\left(\int_\alpha^1 \left(\frac{1}{p}\mathbb{E}_p[N(A_n)|A_n] - \frac{2}{2-p}\mathbb{E}_p[N^+(A_n)|A_n]\right) dp\right)}$$

Moreover, since we are interested in what happens to $P_p(A_n^c)$ as we expand the size of our box for a given p, let's look at the limit of both sides as $n \to \infty$. In applying the limit, we are left with the following theorem.

Theorem 3.2.

$$\lim_{n\to\infty}[1-P_\alpha(A_n^c)] = \lim_{n\to\infty} \frac{[1-P_1(A_n^c)]}{exp\left(\int_\alpha^1 \left(\frac{1}{p}\mathbb{E}_p[N(A_n)|A_n] - \frac{2}{2-p}\mathbb{E}_p[N^+(A_n)|A_n]\right) dp\right)}\ .$$

The above theorem is a very interesting and important result because we can easily derive a simplified restatement of the recurrence question for $0 < p < 1$: *If we can show that the right hand side of Theorem 3.2 goes to* 0*, it follows that*

$$\lim_{n\to\infty}[1-P_\alpha(A_n^c)] = 0$$

for any $0 < \alpha < 1$*, or alternatively that*

$$\lim_{n\to\infty} P_\alpha(A_n^c) = 1,$$

which is, of course, the case of recurrence for all $0 < p < 1$.

We know that the limit of the numerator goes to 0. In addition, we have strong indications that the limit of the denominator also goes to 0 (see figure 5). So, it remains to be shown that the numerator goes to zero faster than the denominator. Thus, it will be the focus of the next section to analyze the relative rates of convergence of the limits of the numerator and denominator. We will discuss the necessary conditions for the limit of this quotient to go to 0, and narrow our analysis down to two cases.

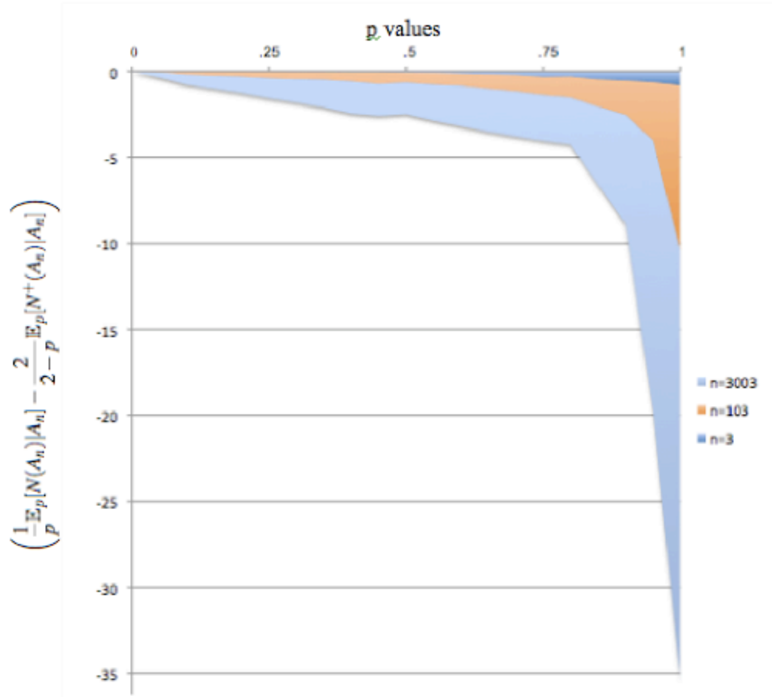

FIGURE 5. Approximate area values inside of exponential as n increases

4. Relative Limits

It will be the focus of this section to analyze the relative rates of convergence to zero of the respective limits of the numerator and denominator of the right hand side of Theorem 3.2. It is our goal to show that the limit of the entire right hand side goes to zero; that is, to prove the numerator goes to zero *faster* than the denominator. In our attempts to do so, we have discovered and proven the following theorem:

Theorem 4.1. *Either the* $\lim_{n\to\infty} P_p(A_n) = 0$ *for all* $0 < \alpha \le p < 1$*, OR for some interval of probabilities* (p_1, p_2)*, where* $0 < \alpha \le p_1 < p_2 \le 1$*, in order for two complimentary paths from the origin to be infinite - the event* A_∞ *- there are necessarily infinitely many pivotal*$^{\pm}$ *points for that event.*

The proof of Theorem 4.1 follows from the next two subsections. We will provide some commentary along with the proof to help explain some of the more crucial and less obvious steps.

4.1. **Limit of the Numerator.** Using techniques from [1] we can analyze the limit as $n \to \infty$ of the numerator from Theorem 3.2 to figure out exactly how quickly it converges to zero. We know from this book that

$$\begin{aligned} [1-P_1(A_n^c)] &\leq \sum_{k=n}^{\infty} k^{-(1+\frac{1}{\rho})(1+\varepsilon)} \\ &\approx \int_n^{\infty} x^{-(1+\frac{1}{\rho})(1+\varepsilon)} dx \\ &= \frac{\rho}{1+\varepsilon+\rho*\varepsilon} * n^{-\frac{(1+\varepsilon+\rho*\varepsilon)}{\rho}} \end{aligned}$$

for any sufficiently small $\varepsilon > 0$ and $\rho \geq 3$ (The basis for this argument is found in [1], on pages 236 and 279).

As a result, we know that the probability, in the case of $p = 1$, that our light beam escapes D_n goes to zero at least as quickly as the right hand side of the above equation. That is,

$$[1-P_1(A_n^c)] \approx \frac{\rho}{1+\varepsilon+\rho*\varepsilon} * n^{-\frac{(1+\varepsilon+\rho*\varepsilon)}{\rho}} . \tag{5}$$

4.2. **Limit of the Denominator.** Now, let's take a look at the limit as $n \to \infty$ of the denominator of Theorem 3.2. Since p and $E_p[N(A_n)|A_n]$ are always greater than or equal to zero, we know that

$$\lim_{n\to\infty} exp\left(\int_\alpha^1 \left(\frac{1}{p}\mathbb{E}_p[N(A_n)|A_n] - \frac{2}{2-p}\mathbb{E}_p[N^+(A_n)|A_n]\right) dp\right) \geq \lim_{n\to\infty} exp\left(-\int_\alpha^1 \frac{2}{2-p}\mathbb{E}_p[N^+(A_n)|A_n]dp\right) .$$

Furthermore, since $2 \geq \frac{2}{2-p}$ for all p,

$$\lim_{n\to\infty} exp\left(-\int_\alpha^1 \frac{2}{2-p}\mathbb{E}_p[N^+(A_n)|A_n]dp\right) \geq \lim_{n\to\infty} exp\left(-2*\int_\alpha^1 \mathbb{E}_p[N^+(A_n)|A_n]dp\right) .$$

Then obviously,

$$\lim_{n\to\infty} exp\left(\int_\alpha^1 \left(\frac{1}{p}\mathbb{E}_p[N(A_n)|A_n] - \frac{2}{2-p}\mathbb{E}_p[N^+(A_n)|A_n]\right) dp\right) \geq \lim_{n\to\infty} exp\left(-2*\int_\alpha^1 \mathbb{E}_p[N^+(A_n)|A_n]dp\right) ,$$

which means that the limit of the denominator of Theorem 3.2 cannot converge to zero any faster than

$$\lim_{n\to\infty} exp\left(-2*\int_\alpha^1 \mathbb{E}_p[N^+(A_n)|A_n]dp\right) .$$

We want to show that this limit converges to zero more slowly than $\frac{\rho}{1+\varepsilon+\rho*\varepsilon} * n^{-\frac{(1+\varepsilon+\rho*\varepsilon)}{\rho}}$. If this is the case, then we certainly have recurrence for all $0 < \alpha \leq p < 1$. So, for the limit of the denominator to go to zero more slowly than the numerator we need that

$$exp\left(-2*\int_\alpha^1 \mathbb{E}_p[N^+(A_n)|A_n]dp\right) > \frac{\rho}{1+\varepsilon+\rho*\varepsilon} * n^{-\frac{(1+\varepsilon+\rho*\varepsilon)}{\rho}} * (n^\varepsilon)$$

for all n, where for any fixed $\varepsilon > 0$, the $\lim_{n\to\infty} n^{\varepsilon} = \infty$. Now, let's simplify this equation a little further:

$$\begin{aligned}\left(-2 * \int_{\alpha}^{1} \mathbb{E}_p[N^{+}(A_n)|A_n]dp\right) &> \ln(\rho) - \ln(1+\varepsilon+\rho*\varepsilon) - \frac{1+\varepsilon+\rho*\varepsilon}{\rho}\ln(n) + \varepsilon\ln(n) \\ &> -\frac{1+\varepsilon+\rho*\varepsilon}{\rho}\ln(n) + \varepsilon\ln(n)\end{aligned}$$

Due to the fact that $E_p[N^{+}(A_n)|A_n] = E_p[N^{-}(A_n)|A_n]$ for all p and n, we can make the following adjustment:

$$\left(\int_{\alpha}^{1} \mathbb{E}_p[N^{\pm}(A_n)|A_n]dp\right) < \left(\frac{1+\varepsilon+\rho*\varepsilon}{\rho}\ln(n) - \varepsilon\ln(n)\right). \tag{6}$$

Obviously, if the above inequality holds, then the right hand side of Theorem 3.2 certainly goes to zero, and we have recurrence for all $0 < \alpha \le p < 1$. But what happens if the above inequality does not hold? Let's assume

$$\left(\int_{\alpha}^{1} \mathbb{E}_p[N^{\pm}(A_n)|A_n]dp\right) \ge \left(\frac{1+\varepsilon+\rho*\varepsilon}{\rho}\ln(n) - \varepsilon\ln(n)\right) = \left(\frac{1+\varepsilon}{\rho}\ln(n)\right). \tag{7}$$

So, just to be clear, either Equation 6 or Equation 7 is true; there is no other option. From Equation 7, we know that for at least some interval (p_1, p_2) where $0 < \alpha \le p_1 < p_2 \le 1$,

$$\lim_{n\to\infty}\left(\mathbb{E}_p[N^{\pm}(A_n)|A_n]\right) \ge \lim_{n\to\infty}\left(\frac{1+\varepsilon}{\rho(1-\alpha)}\ln(n)\right) = \infty. \tag{8}$$

This equation tells us that since the limit as $n \to \infty$ of the right hand side is ∞, the limit of the left hand side must also be infinite. So in order for two complementary paths from the origin to visit infinitely many vertices and remain unbounded, there must be infinitely many pivotal$^{\pm}$ points for that event. Essentially what this means is that for two paths to remain independent forever, they must intersect each other infinitely many times. At this point, we have arrived at Theorem 4.1.

4.3. **Informal Explanation and Conjecture for Case 2 of Theorem 4.1.** This is a very interesting, and admittedly confusing result. What Theorem 4.1 essentially tells us is that, for some values of p and very large n, the expectation of pivotal$^{\pm}$ points has to be proportionally large (on the order of $\ln(n)$). This is somewhat counterintuitive given that, if two paths from the origin both reach the edge of the boundary, we expect those paths to intersect each other very rarely, because each time they intersect there is a probability (> 0) that the path will be closed off at that vertex. The probability that the path will be closed off at any given pivotal point is at the very least $min\{(1-p), \frac{p}{2}\}$, because there is always exactly one mirror (or no mirror) that will close the circuit when two paths intersect; see Figure 6. So, with this in mind, it does not make sense that two paths that reach the edge of a boundary of arbitrary size (or even infinite in this case) would intersect several (infinitely many) times.

After studying this question for quite some time, it becomes apparent that we should almost certainly be able to derive a contradiction from Equation 8, and use it to our advantage. We can express the probability that we do not form a loop in a finite region D_n in a very specific way. In order for two paths to reach the edge of a finite box D_n, we know that each time they meet, they must avoid the one situation that will close the loop (there is always exactly one: either a

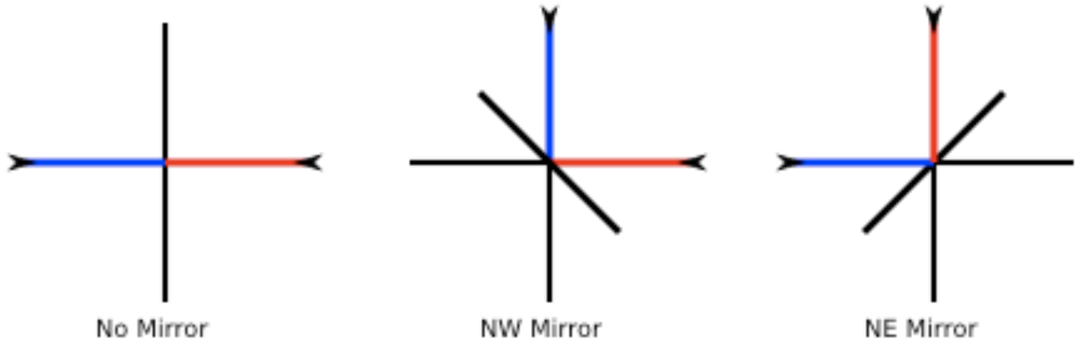

FIGURE 6. The Three Cases for Path Closure

NW mirror, NE mirror, or no mirror). So, at any given vertex of intersection there is at most a probability of $\left(1 - min\{(1-p), \frac{p}{2}\}\right)$ that the path *will not* be closed. Obviously, the probability of this event occurring is less than one for all $0 < p < 1$. All the information we have obtained pertains to pivotal$^{\pm}$ points and we know that the probability of two paths continuing on from a pivotal$^{\pm}$ vertex is still at most $\left(1 - min\{(1-p), \frac{p}{2}\}\right)$. Thus, we can express an approximate probability that the two independent paths from the origin will reach the boundary of our box D_n as the inequality:

$$P_p(A_n) \leq k * \left(1 - min\{(1-p), \frac{p}{2}\}\right)^{\mathbb{E}_p[N^{\pm}(A_n)|A_n]} ,$$

for some constant, k, which depends upon the orientation of the rest of the lattice. However, according to Equation 8, our expectation of the number of pivotal$^{\pm}$ points is increasing to the order of $\ln(n)$ which goes to infinity. Thus, since we are raising something less than one to an infinite power, the probability that two paths continue on forever without ever forming a closed loop is:

$$\lim_{n\to\infty} P_p(A_n) \leq \lim_{n\to\infty} k * \left(1 - min\{(1-p), \frac{p}{2}\}\right)^{\mathbb{E}_p[N^{\pm}(A_n)|A_n]} = 0 . \tag{9}$$

While the above argument is not stringent enough to deserve its own theorem, one can easily see that:

Conjecture 4.2. *For every α, there exists at least some interval, (p_1, p_2) where $0 < \alpha \leq p_1 < p_2 \leq 1$ such that*

$$\lim_{n\to\infty} P_p(A_n) = 0$$

for all $p \in (p_1, p_2)$, which is the case of recurrence for all such values of p.

Of course, this is essentially a conjecture inside of a conjecture, but we feel that Conjecture 4.2 is much more tangible than the main recurrence hypothesis. What this argument is lacking is a way to tie $P_p(A_n)$ to the Borel-Cantelli lemma. We feel that it is a very short jump from this point to the lemma, which would show that the probability that two paths intersect at pivotal$^{\pm}$ points infinitely many times is equal to zero. This, of course, is a contradiction of case two, which would leave us with only case one in Theorem 4.1. It remains to be shown that

$$\sum_{N}^{\infty} P_p(N^{th} \text{ pivotal}^{\pm} \text{ point occuring}) < \infty \,,$$

which will most likely involve a similar argument to the one outlined in this section.

5. Conclusion and Future Work

At this point, the recurrence hypothesis remains relatively wide open. However, we have broken the hypothesis down into two possible cases: one that proves recurrence; and another that tells us that in order for some complementary paths from the origin to be infinite, they must intersect at pivotal$^{\pm}$ points infinitely often, from which a contradiction can almost certainly be derived.

We are continuing work on this problem, and presently we will be attempting to derive the aforementioned contradiction. It has proven relatively difficult to apply the Borel-Cantelli lemma to this particular problem, but we are confident that there exists some sort of argument that will allow us to apply the lemma and prove recurrence for all $0 < p < 1$.

There is also another direction that could yield results if explored. If one could show that two complimentary paths from the origin are guaranteed to meet *even once* at the same time step, recurrence would follow almost directly. This is due to the fact that there is a strictly positive probability that the loop will be closed off at that point, and even if the two paths continue on independently, they would be guaranteed to meet at the same time step again and again until the path was finally closed off.

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