Problem statement: what are wells
Numerical solution of idealized model
Peaceman correction and beyond
Idealized (with point sources) versus real model

Peaceman and Thiem well models
or
how to remove a logarithmic singularity from your numerical solution

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Outline

1. Problem statement: what are wells
   - Steady state, rate specified wells
   - Idealized and real well models

2. Numerical solution of idealized model
   - Not enough regularity for classical FE ...
   - Why subtracting singularities is not sufficient

3. Peaceman correction and beyond
   - Effective well radius idea: Peaceman correction
   - Beyond Peaceman correction

4. Idealized (with point sources) versus real model
   - Closeness of solutions: elliptic and parabolic estimates
What are wells

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Peaceman and Thiem well models or how to remove a logarithmic singularity from your numerical solution
Well in a confined aquifer/hydrocarbon reservoir
Pressure profiles around the well: side view
Pressure profiles around the well: aerial view
Mathematical model: geometry

\[ \Omega \]

\[ \Omega_w \]

\[ r \]

\[ \delta \Omega \]
Problem statement for steady state, rate specified, $\mathbb{R}^2$

Conservation equations:

$$\mathbf{u} = -K \nabla p, \quad x \in \Omega \quad \text{Darcy's law}$$

$$\nabla \cdot \mathbf{u} = \bar{q}, \quad \text{conservation of mass}$$

$ar{q}(x)$ represents source/sink terms, and is known,

$K$ is the conductivity tensor and is known.

Put it together

$$-\nabla \cdot K \nabla p = \bar{q}(x), \quad x \in \Omega$$

$$p|_{\partial \Omega} = p_{\text{GIVEN}} = 0$$
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Mathematical model for wells: idealized and real

Geometry $\Omega_w \approx B(x_0, r_w)$ with boundary $\Gamma_w$

1. (R) Real model of injection/production wells

$$- \nabla \cdot K \nabla p = 0, \quad x \in \Omega \setminus \Omega_w$$

$$K \nabla p \cdot \nu = \frac{q}{2\pi r_w}, \quad x \in \partial \Omega_w$$

2. (ID) Idealization of a “true” real-life situation

$$- \nabla (K \nabla p^\delta) = \bar{q}(x) = q \delta_{x_0}(x), \quad x \in \Omega$$

where by $\delta_{x_0}(x)$ we mean the Dirac-$\delta$ distribution

- $\delta_{x_0}(x) = 0, \quad x \neq x_0$
- $\int_D \delta_{x_0}(x) = 1,$

and where $q$ (total rate) is given as data.
Solution to the mathematically idealized problem (ID)

Analytical solution in 2D, steady state, with one well at 0, single-phase flow with homogeneous isotropic $K = K I$

\[-K \nabla \cdot (\nabla p^\delta) = q_0 \delta, x \in \Omega\]

Change variable to polar coordinates $p^\delta = p^\delta(r, \theta)$, assume radial solution $p^\delta = p^\delta(r)$:

\[-\frac{1}{r} \frac{\partial}{\partial r} (Kr \frac{\partial p^\delta}{\partial r}) = q_0 \delta, r > 0\]

Solution has a logarithmic singularity

$p^\delta(r) = C_1 \log(r) + C_2, r > 0$

Fix the constants (use total input $q$ and boundary conditions on $\partial \Omega$

$C_1 = \frac{q}{2\pi K}, C_2 = p_{\text{GIVEN}}$ (easiest case $C_2 = 0$) to get

$p^\delta(r) = \frac{q}{2\pi K} \log(r)$
Analytical solution to (ID) is not very useful ...

when (bottom hole) pressure in the well is needed

Problem !!!! $p^\delta(x_0) = \infty$

when multiple wells are present:

$$-\nabla \cdot K \nabla p = \sum_i q_i \delta_{x_i}, \quad x \in \Omega$$
Solution to the “real” single well problem (R)

Known: $K, r_w, q$. Solve

\[-\nabla \cdot K \nabla p = 0, \quad x \in \Omega \setminus \Omega_w\]
\[K \nabla p \cdot \nu = \frac{q}{2\pi r_w}, \quad x \in \partial \Omega_w\]

Similarly as in (ID) model we can derive

\[p(r) - p(R) = \frac{q}{2\pi K} \log\left(\frac{r}{R}\right)\]

where $p(R)$ is given at some $R > 0$.

For example, $p(R)$ can be given at some distance away from wells (Thiem solution) \cite{Guenther,Lee 8.1/#12,Marsilly’86}

Analytical solution still not useful when multiple wells are present
Try numerical solution for (R) real well model ...

Very fine grid around (multiple) wells would be necessary.

Still no good way to get $p(r_w)$
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Numerical handling of idealized model using FE

Functional spaces ... notation

- \( W^{m,p}(\Omega) := \{ u : \partial^m u \in L^p(\Omega) \} \)
- \( W^{0,2}(\Omega) \equiv L^2(\Omega) \)
- \( W^{m,2}(\Omega) \equiv H^m(\Omega) \)

Assume smooth \( \bar{q} \) for the moment ... derive weak form of

\[-\nabla (K \nabla p) = \bar{q} \]

\[ \int_{\Omega} K \nabla p \nabla w = \int_{\Omega} \bar{q} w, \; \forall w \in H^1_0(\Omega) \]

(Classical FE Galerkin method:) Find \( p_h \in V_h \) which approximates \( p \) and solves:

\[ \int_{\Omega} K \nabla p_h \nabla w_h = \int_{\Omega} \bar{q} w_h, \; \forall w_h \in V_h \subsetneq H^1_0(\Omega) \]

Error estimate \( \| p - p_h \|_{H^1(\Omega)} \leq C h \| p \|_{H^2(\Omega)} \)

But \( \bar{q} \) is not smooth ... and \( p \) is not smooth ...
How nonsmooth are solutions to (ID) ...?

Elliptic theory [GilTru, Dautray-Lions] tells us that weak solution to

$$\nabla(K\nabla p) = \bar{q}$$

has the following regularity

- $\bar{q} \in L^2(\Omega) = W^{0,2}(\Omega) \implies p \in H^2(\Omega)$
- $\bar{q} \in H^{-1}(\Omega) = W^{-1,2}(\Omega) \implies p \in H^1(\Omega)$
- $\bar{q} \in L^1(\Omega) = W^{0,1}(\Omega) \subset W^{-1,p}(\Omega) \implies p \in W^{1,p}(\Omega)$ ($p < 2$).

But $\delta_{x_0} \notin L^p(\Omega)$ !!

Note: $\int_\Omega \delta_{x_0} w = w(x_0)$ requires $w \in C^0(\Omega)$ or that

$\delta_0 \in (C^0(\Omega))' = M(\Omega)$ (dual space to $C^0(\Omega)$).

However, we have $W^{1,p'}(\Omega) \subset C^0(\bar{\Omega})$ so

$\bar{q} \in (C^0(\bar{\Omega}))' \equiv M(\Omega) \subset W^{-1,p}(\Omega)$ if $p < 2$ (Sobolev imbedding Thm)

hence the solution $p \in W^{1,p}(\Omega)$ for any $p < 2$.

This is not enough regularity for FE solution to converge
We do not have $p \in H^1(\Omega)$, hence, $p_h$ may fail to converge to $p$.

Idea: subtract the singular part of the solution

Consider a general problem $-\nabla \cdot K \nabla p = \sum_i q_i \delta_{x_i}, \ x \in \Omega$.

Construct

$$p^S(x) = \sum_i \frac{q_i}{2\pi K_i} \log(r_i(x))$$

and instead of $p$ approximate $p^{\text{REGULAR}} := p - p^S$, that is, solve

$$\int_{\Omega} K \nabla p_h^{\text{REGULAR}} \cdot \nabla w_h = -\sum_i \int_{\partial \Omega} K \nabla p^S \cdot \nu w_h, \ \forall w_h \in V_h$$

and so $p^{\text{REGULAR}} \in H^2(\Omega)$ (K homogeneous).

If $K_i$ are different, then one can show $p^{\text{REGULAR}} \in H^{2-\epsilon}(\Omega)$, and almost optimal convergence rates for $p^{\text{REGULAR}} - p_h^{\text{REGULAR}}$ are
The above idea (solve for \((p^{\text{REGULAR}})_h\), add \(p^S\)) works well at a sufficiently large distance from wells. However, at wells it is STILL designed to give \(p(x_i) = \infty\).

- good for rate-specified wells (when \(q\) is given) when \(p(x_i)\) is not needed
- not applicable with pressure-specified wells (\(p(x_i)\) is given)

Idea: consider the idea of an “effective well radius” … where we exploit the familiar equation from (R) real well model

\[
p(r) - p(r_w) = \frac{q}{2\pi K} \log\left(\frac{r}{r_w}\right)
\]
Knowing \( q \), how to get \( p_w \)?

Given \( q \), we can hypothetically compute a numerical solution

\[ p_h \equiv p = (p_0, p_1, \ldots p_N) \text{ where } p_0 \approx p(x_0) \]

- we have a numerical method in which we solve for \( p_h \) using \( q \)

\[ Ap = f \]

but clearly \( p_w \neq p_0 \) !!!

- but ... we have an analytical formula

\[ p(r) - p(r_w) = \frac{q}{2\pi K} \log\left(\frac{r}{r_w}\right) \]

we hope this formula applies already to \( p_1, 2, 3, 4 \)

Idea: combine these two methods to get an expression for \( p_w \)
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Effective well radius idea: Peaceman correction
Beyond Peaceman correction

Peaceman correction [Peaceman’77] (homogeneous isotropic $K$, isotropic grid)

Cell Centered Finite Difference $\approx$ Mixed Finite Element with RT0

1. \[ -\frac{1}{h} \left( K \frac{p_1 - p_0}{h} + K \frac{p_2 - p_0}{h} + K \frac{p_3 - p_0}{h} + K \frac{p_4 - p_0}{h} \right) = \frac{q}{h^2} \]

2. \[ p_{1,2,3,4} - p_0 = \frac{q}{2\pi K} \log \left( \frac{h}{r_w} \right) \]

Combining 1,2 we get

\[ p_0 = p_w + \frac{q}{2\pi K} \left( \log \left( \frac{h}{r_w} \right) - \frac{\pi}{2} \right) = p_w + \frac{q}{2\pi K} \left( \log \left( \frac{r_0}{r_w} \right) \right) \]

where \( r_0 = \exp(-\pi/2)h \approx 0.208h \) is the effective well radius.
But what if $K$ is not isotropic homogeneous?

$K = K^T$ (permeability, conductivity, mobility,...) is in general anisotropic and heterogeneous. And grids are not uniform and isotropic ...

- Peaceman models allow for $K = \text{diag}(K_{11}, K_{22}), h_x \neq h_y$
- homogenization for Darcy flow around wells [Zijl, Trykozko’01]
- multiscale FE method for Darcy flow near wells [Z. Chen et al ’03]
- extension to non-Darcy flow, $K \equiv \text{const}$ [Lazarov, Ewing et al’99]
- CG & MP ... in progress
Results: Darcy with rate-wells and BHP wells
Idealized model versus real model

- Good in many situations when
  \[\text{diam}(\Omega) \gg \text{diam}(\Omega_w)\]

But...

- How close is \(p^\delta\) to the true \(p\)?
- What if there are more wells and flow is not single phase but multiphase?
  - Use numerical solution: but how to guarantee convergence?
- We need to know \(p(r_w)\) (pressure in well)! in order to determine phase behavior
- What if \(K\) is not constant and not isotropic?
How close is $p^\delta$ to $p$?

For non-stationary version of the model, as $r_w \to 0$, one has (if coefficients are extended properly) [G. Chavent, J. Jaffré ’86]

$$\| p^\delta - p \|_{L^2(\Omega \times (0, T))} \to 0$$

For the original stationary problem:
[Li Ta Tsien, A. Damlamian’80], [Da Qian Li, Shu Xing Chen (in Chinese)’78],
Recent result [Zhiming Chen, Xinye Yue’03]

$$\max_{x \in \Omega} |p^\delta - p| \leq Cr_w |\log(r_w)|$$
[Zhiming Chen, Xinye Yue’03]
Assume: $K \in C^{0,1}(\Omega)$ with Lipschitz constant $\Lambda$
Thus for any $p > 0$ the solution satisfies

$$(p^\delta - p) \in W^{2,p}(\Omega) \implies (p^\delta - p) \in C^1(\overline{\Omega})$$

and we have the following estimate

$$\max_{x \in \Omega} |p^\delta - p| \leq C\Lambda (1 + \log(\Lambda))^{1/2} r_w |\log(r_w)|$$
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Closeness of solutions: elliptic and parabolic estimates