MTH XXX Homework 0. **FirstName LastName**
Tips how to write the solutions. Tips on this HW. Date: December 29, 2017
Prof. M. Peszynska Due: 02/13/1976

Problem 1. Discuss the error in the approximation of $f'(x)$ by the difference quotient $D_h f := \frac{f(x+h)-f(x)}{h}$. Include the mathematical details. Be brief.

Solution: We assume f is C^2 smooth in some interval $x \in [a,b]$ including $(x, x+h)$. We expand $f(x+h)$ with Taylor polynomial with the remainder in Lagrange form

$$(1) \qquad f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(\xi) \ \ \xi \in (x, x+h).$$

(Quote your sources. For example: Here we follow the textbook by Fitzpatrick [1, p.203].)
We see then that

$$(2) \qquad D_h f(x) = f'(x) + \frac{h}{2}f''(\xi).$$

Since f'' is bounded on $(x, x+h)$, the error $e(h) = |D_h f(x) - f'(x)| = O(h)$ is <u>of first order in h</u>.

Problem 2. Confirm the findings in Problem 1 with numerical experiments for $f(x) = \frac{1}{1+x^2}$. The order α of convergence in $e(h) \approx h^\alpha$ may not always be known a-priori. Find and confirm the order experimentally, i.e., using plots, tables, and curve fitting.
Extra: *To explore roundoff error issues, take $h \approx 10^{-9}$ or closer to machine precision.*

Solution: We implement code in MATLAB to find the error. Show as much code as necessary but not more. Don't dwell on the obvious. Annotate with comments to clarify. Mathematical details do not belong in the code comments.

Code I.
```
%% define function and its derivative
f=@(x)(1./(1+x.^2));  fd=@(x)(-2*x./(1+x.^2).^2);  x0=1;
%%
j=1;h(j)=10^(-j);hh=h(j);e(j)=abs((f(x0+hh)-f(x0))/hh-fd(x0))
j=2;h(j)=10^(-j);hh=h(j);e(j)=abs(f(x0+hh)-f(x0))/hh-fd(x0))
...
%%
loglog(h,h,'k-',h,e,'r*-');grid on
title('Error of D_hf');legend('linear','e(h)');
print -dpng errorh.png
```

Always refer properly to the code, figures, and tables. Figure 1 shows the convergence plot generated by Code I which confirms the first order of convergence. We determine the order of convergence with Code II.
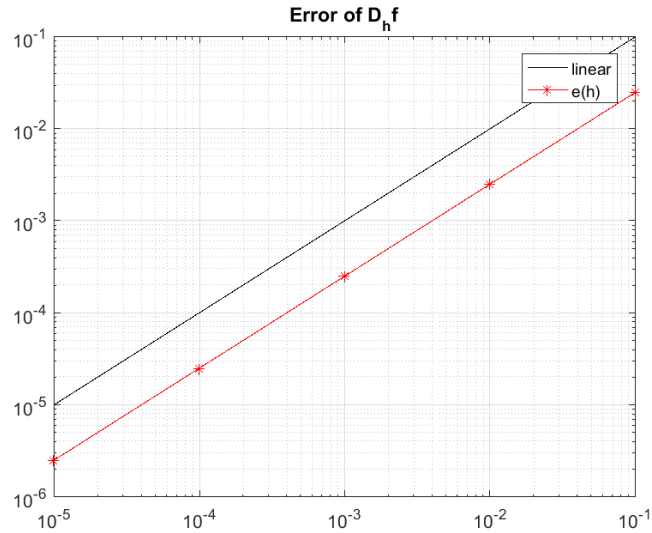
FIGURE 1. Error behavior. The error $e(h)$ appears to have the same slope as the linear trend. The plot should have a legend, title, and labels. Always use markers in addition to colors.

Code II.

```
%% assuming you have the vectors h,e of length 5
%% calculate the order from the error
for j=2:5
alpha(j)=(log(e(j))-log(e(j-1)))/(log(h(j))-log(h(j-1)));
end
format long; alpha
%% you can also confirm the order by using polyfit
polyfit(h,e,2)
```

h	$e(h)$	α
0.1	0.024886877828054	
0.01	0.002499876243744	0.998051906332859
0.001	0.000249999875134	0.999978717718038
0.0005	... include more data	

TABLE 1. Error behavior. When h decreases by factor 10, so does the error. Matching $e(h)$ to h^α reveals that $\alpha \approx 1$. Choose a variety of h

The result in Table 1 shows again that the error $e(h)$ decreases linearly with h. The polynomial fit shows that the dominant terms are of a linear function $e(h) \approx 0.25h - 0.012h^2 = O(h)$. Include proper citations to any sources you used including software.

REFERENCES

[1] P.M.Fitzpatrick *Advanced Calculus*. Amarican Mathematical Society, 2006.
[2] CTAN archive of the LaTeX package `listings` https://ctan.org/pkg/listings