MTH 453-553 W2013, Assignment 3

Students registered for 453 solve 1a, 2a. Then 3 and 4ac (or more for extra credit).

Students registered for 553 solve 1ab, 2ab. Next 3 and 4 (or more for extra credit).

Theoretical part is due before the Midterm.

1. Consider solving diffusion-reaction equation $u_t - Du_{xx} + Ru = f$ on $(0,1)\times(0,1)$, where the diffusion constant $D > 0$ and reaction constant R are given, with the scheme

$$\frac{u_j^{n+1} - u_j^n}{k} + D\frac{2u_j^{n+1} - u_{j-1}^{n+1} - u_{j+1}^{n+1}}{h^2} + R\frac{u_{j-1}^n + u_{j+1}^n}{2} = f(x_j, t_{n+1}), \quad (1)$$

which, if $R = 0$, is the same as BE, but for $R \neq 0$ it approximates the reaction term explicitly in time in an unusual way.

a) Calculate the local truncation error for the scheme (1) to show it is $O(k + h^2)$.

b) Repeat a) for at least one scheme from the following more "usual" schemes: implicit-explicit schemes

$$\frac{u_j^{n+1} - u_j^n}{k} + D\frac{2u_j^{n+1} - u_{j-1}^{n+1} - u_{j+1}^{n+1}}{h^2} + Ru_j^n = f(x_j, t_n), \quad (2)$$

$$\frac{u_j^{n+1} - u_j^n}{k} + D\frac{2u_j^n - u_{j-1}^n - u_{j+1}^n}{h^2} + Ru_j^{n+1} = f(x_j, t_n), \quad (3)$$

a scheme that is entirely explicit

$$\frac{u_j^{n+1} - u_j^n}{k} + D\frac{2u_j^n - u_{j-1}^n - u_{j+1}^n}{h^2} + Ru_j^n = f(x_j, t_n), \quad (4)$$

or entirely implicit

$$\frac{u_j^{n+1} - u_j^n}{k} + D\frac{2u_j^{n+1} - u_{j-1}^{n+1} - u_{j+1}^{n+1}}{h^2} + Ru_j^{n+1} = f(x_j, t_{n+1}). \quad (5)$$

2. If $D > 0, R \geq 0$ in Pbm 1, then the true solution should decay when $f \equiv 0$. a) Analyze the stability of the schemes (2), (5) using von-Neumann analysis. Find sufficient conditions on k to make sure the magnification factor is ≤ 1, i.e., so that the numerical solution decays as well.

b) Consider also the stability of the schemes (4), (3).

Extra: analyze all of (1)-(5).

Computational part is due after the Midterm.

3. Modify the code `fd1d_heat.m` and implement a) FE scheme, b) Crank-Nicolson scheme for $u_t - Du_{xx} = f$, or both.
   Confirm their stability and convergence as predicted by theory, when $D = 1, 10, 1/10$. (You can use the same true solution as that already in the code, but recompute f.)
   That is, show the unconditional stability of the CN scheme and only a conditional stability of FE. For the latter, show that CN gives reasonable solutions no matter how large k is, but FE blows up unless the stability condition holds.
   Extra: Compare the computational effort theoretically and experimentally (with `tic,toc`) when you choose a time step for optimal convergence so that the global error would be under 10^{-4}.

4. a) Implement the scheme (5) and confirm that, as expected, it is $O(k+h^2)$-order convergent and unconditionally stable. Use $D = 1$ and $R = 1$, $R = 10$, $R = 1/10$. Assume $u = sin(\pi x)e^{-t}$ as the true solution. (Compute the appropriate f and boundary conditions).
   b) Do the same as a) for at least one of the other schemes. Compare.
   c) Now assume you do not know the solution. Find the numerical solution when $f(x,t) \equiv 1$ either with (5), or (1). Is there a stationary limit $u_\infty(x) = \lim_{t\to\infty} u(x,t)$? Comment on the physical behavior of the solutions depending on R (Use $D = 1$, and $R = 0, 1, 10, 1/10$).
   Extra: play with boundary conditions other than homogeneous Dirichlet. Also, play with different values of D and consider the behavior depending on the ratios D/R.

In Pbm 4c you should change the "limits" in the plot to make sure you "catch" the stationary limit. Plot all the $u_{\text{inf}}(x)$ corresponding to different R on the same plot. Use a reasonable h and k in the experiments.