MTH 654/659 Fall 2011. Assignment 2

Instructions: Please solve 3 problems out of 1-4 or more for extra credit. In 4, choose A, B, or C, or solve more for extra credit. Do not turn in 5 (yet).

Group work is allowed on computational examples but you have to write/type all your solutions yourself. In computational assignments in 1D, you are welcome to use the code `fem1d.m` posted at
`http://www.math.oregonstate.edu/~mpesz/code/cmatlab/mth655/fem1d.m`
In 2D, please use either ACF code or DEAL II.
When reporting on your computational experiments, provide a concise summary of what you have observed/determined. [Do not just staple a bunch of graphs and papers together].

Problems

1. [**Analysis**] Propose an example where the solution to $-u'' = f$ with homogeneous Dirichlet data satisfies $u \in H^3(\Omega) \cap H_0^1(\Omega)$ but $u \notin H^4$. What V_h should you use to get the optimal order FE solution for that problem ? (Please use proper vocabulary)
   [**Computation**] Implement (ii) by modifying FEM1D.m and demonstrate that the convergence holds for the optimal case; compare with some non-optimal choices of V_h.
2. Use FE with linear elements and a uniform triangular or rectangular grid to solve the problem $-\triangle u = f$ on $(0,1)^2$, with homogeneous boundary conditions so that $u = sin(\pi x)y(1-y)$ is the exact solution. Confirm the expected convergence rate in L^2, H^1 norm. **Extra:** Verify the L^∞ convergence estimate given on p.93 (Eq.II.7.8) in the textbook.
3. Consider the problem $-\nabla \cdot (\mathbf{K}\nabla u) = f$ on $\Omega = (-1,1)\times(-1,1)$ with Dirichlet boundary conditions of your choice, and where $\mathbf{K} = \mathbf{K}(x,y)$ is a given diagonal tensor. Assume the exact solution is given by $u(x,y) = xt(y)$ where
   (i) $t(y) = y$, $\mathbf{K} = 2\mathbf{I}$
   (ii) $t(y) = y$, $\mathbf{K} = diag(1,2)$
   (iii) $t(y) = \begin{cases} y, & y < 0 \\ 10y & y \geq 0 \end{cases}$, $\mathbf{K} = \mathbf{I}$
   (iv) $t(y) = \begin{cases} y, & y < 0 \\ 10y & y \geq 0 \end{cases}$ $\mathbf{K}(x,y) = \begin{cases} 10\mathbf{I}, & y < 0 \\ \mathbf{I} & y \geq 0 \end{cases}$
   [**Analysis**] For which $t, \mathbf{K}$ are the problem and the solution regular enough so that all desirable FE error estimates hold ?
   [**Computation**] Make appropriate modifications to the ACF code to solve the problem. Confirm the convergence rate you expected from Analysis part.
   Should Dirac δ appear (in the right hand side) and you need help with implementing it, ask me.
4. Choose one of [A-C]:
   [**A**] Consider (in 2D, for triangles) the affine transformation F from the reference element $\hat{T}$ to a "real element" T with nodes (x_1, y_1).... Compute the local stiffness matrix a_T using the map F and directly by finding the basis functions and their derivatives (results should be, of course, the same).

Relate your computation to the calculations in ACF code for triangles.

[**B**] Plot images of a reference triangle $\hat{T}$ under i) some nontrivial affine map of your choice and ii) an isoparametric map that turns the hypotenuse of the reference triangle into an arc of the unit circle (you need to derive it first). For the latter, consider the inverse map and calculate the basis functions on T.

[**C**] Convince yourself of the degree to which *variational crimes* are allowed in FEM. Use theory from First Strang lemma (p. 106 in the textbook) as background. Compare the solutions to $-u'' + u = x$ using numerical quadrature to those with exact integrals to get the matrix and the right hand side of the linear system; include the consideration of static condensation and a comparison with a finite difference solution for the same problem.

5. (Open ended, do not turn in yet.) Make sure that the grid for your personal letter is working ... use it in ACF code with some choice of boundary conditions and f.