Homework 3, Real Analysis Due Friday, October 16, 2015.
Assigned exercises from Chapter 1 of the text: (37, 42, 45, 46, 48).
Optional exercises from Chapter 1 of the text: (38*, 39*, 47*, 50*, 51*).
The following exercises are also assigned:

(I1) (a) Let
\[ A_n = \left[ -\frac{1}{n}, 1 + \frac{1}{n^4} \right]. \]
Find lim sup \( A_n \) and lim inf \( A_n \).

(b) Let
\[ B_n = \left( 1 + (-1)^n \frac{1}{n}, 3 + \frac{1}{n^4} \right). \]
Find lim sup \( B_n \) and lim inf \( B_n \).

(c) Let
\[ C_n = \left( -n(-1)^n - n, (-1)^n n \right). \]
Find lim sup \( C_n \) and lim inf \( C_n \).

(I2) (a) Let \( \{a_k : k \geq 1\} \) be a sequence in \( \{0, 1\} \). In other words, \( a_k = 0 \) or \( a_k = 1 \) for each \( k \geq 1 \). Clearly the finite product \( \prod_{k=1}^{n} a_k \) is defined as follows:
\[ \prod_{k=1}^{n} a_k = \begin{cases} 
0 & \text{if } a_k = 0 \text{ for some } k = 1, \ldots, n \\
1 & \text{if } a_k = 1 \text{ for all } k = 1, \ldots, n.
\end{cases} \]
Give a reasonable definition of the infinite product \( \prod_{k=1}^{\infty} a_k \).

(b) Recall the characteristic function of a set \( A \):
\[ \chi_A(\omega) = \begin{cases} 
1 & \text{if } \omega \in A \\
0 & \text{if } \omega \in A^c.
\end{cases} \]
Let \( \{A_k : k \geq 1\} \) be a sequence of sets in some space \( \Omega \). Show that the following equalities hold.

\[
\chi \cap_{k=1}^{n} A_k(\omega) = \prod_{k=1}^{n} \chi_{A_k}(\omega) = \min_{1 \leq k \leq n} \chi_{A_k}(\omega)
\]

\[
\chi \cap_{k=1}^{\infty} A_k(\omega) = \prod_{k=1}^{\infty} \chi_{A_k}(\omega) = \lim_{n \to \infty} \min_{1 \leq k \leq n} \chi_{A_k}(\omega)
\]

(I.3) Prove that for an arbitrary sequence of sets \( \{A_n : n \geq 1\} \) in some space \( \Omega \),

\[
\chi_{\lim sup A_n}(\omega) = \bar{\lim}_{n} \chi_{A_n}(\omega) \quad \text{for all } \omega \in \Omega
\]

and

\[
\chi_{\lim inf A_n}(\omega) = \lim_{n} \chi_{A_n}(\omega) \quad \text{for all } \omega \in \Omega.
\]