1. (Car. 1.3) Establish the following apparently different (but "fancier") characterization of the supremum. Let \( A \) be a nonempty subset of \( \mathbb{R} \) that is bounded above. Prove that \( s = \sup A \) if and only if (i) \( s \) is an upper bound for \( A \), and (ii) for every \( \varepsilon > 0 \), there is an \( a \in A \) such that \( a > s - \varepsilon \). State and prove the corresponding result for the infimum of a nonempty subset of \( \mathbb{R} \) that is bounded below.

Solution:

2. (Car 1.4) Let \( A \) be a nonempty subset of \( \mathbb{R} \) that is bounded above. Show that there is a sequence \((x_n)\) of elements of \( A \) that converges to \( \sup A \).

Solution:

3. (Car 1.6) Prove that every convergent sequence of real numbers is bounded. Moreover, if \((a_n)\) is convergent, show that \( \inf_n a_n \leq \lim_{n \to \infty} a_n \leq \sup_n a_n \).

Solution:

4. (Car 1.7) If \( a < b \), then there is also an irrational \( x \in \mathbb{R} \setminus \mathbb{Q} \) with \( a < x < b \). [Hint: Find an irrational of the form \( \frac{p}{q} \sqrt{2} \).]

Solution:

5. (Car 1.8 - optional) Given \( a < b \), show that there are, in fact, infinitely many distinct rationals between \( a \) and \( b \). The same goes for irrationals, too.

Solution:

6. (Car. 1.10) Let \( a_1 = \sqrt{2} \) and let \( a_{n+1} = \sqrt{2a_n} \) for \( n \geq 1 \). Show that \((a_n)\) converges and find its limit. [Hint: Show that \((a_n)\) is increasing and bounded.]

Solution:

7. (Car. 1.11) Fix \( a > 0 \) and let \( x_1 > \sqrt{a} \). For \( n \geq 1 \), define \( x_{n+1} = \frac{1}{2} \left( x_n + \frac{a}{x_n} \right) \).

Show that \((x_n)\) converges and that \( \lim_{n \to \infty} x_n = \sqrt{a} \).

Solution:

8. (Car 1.12 - optional) Suppose that \( s_1 > s_2 > 0 \) and let \( s_{n+1} = \frac{1}{2} (s_n + s_{n-1}) \) for \( n \geq 2 \). Show that \((s_n)\) converges. [Hint: Show that \((s_{2n-1})\) decreases and \((s_{2n})\) increases.]

Solution:

9. (Car. 1.13) Let \( a_n \geq 0 \) for all \( n \), and let \( s_n = \sum_{i=1}^{n} a_i \). Show that \((s_n)\) converges if and only if \((s_n)\) is bounded.

Recall that a sequence of real numbers \((x_n)\) is said to be Cauchy if, for every \( \varepsilon > 0 \), there is an integer \( N \geq 1 \) such that \( |x_n - x_m| < \varepsilon \) whenever \( n, m \geq N \).

Solution:

10. (Car. 1.14) Prove that a convergent sequence is Cauchy, and that any Cauchy sequence is bounded.

Solution:
11. (Car. 1.15) Show that a Cauchy sequence with a convergent subsequence actually converges.

Solution:

12. (Car. 1.16 - optional)

(a) Why is 0.4999\ldots = .5? 
(b) Write .234234234\ldots as a fraction. 
(c) Precisely which real numbers between 0 and 1 have more than one decimal representation? Explain.

Solution: