1. (Car. 4.3) Some authors say that two metrics $d$ and $\rho$ on a set $M$ are equivalent if they generate the same open sets. Prove this. (Recall that we have defined equivalence to mean that $d$ and $\rho$ generate the same convergent sequences. See Exercise 3.42.)

   Solution:

2. (Car. 4.5) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Show that $\{x : f(x) > 0\}$ is an open subset of $\mathbb{R}$ and that $\{x : f(x) = 0\}$ is a closed subset of $\mathbb{R}$.

   Solution:

3. (Car. 4.19) Show that $\text{diam}(A) = \text{diam}(\bar{A})$.

   Solution:

4. (Car. 4.33) Let $A$ be a subset of $M$. A point $x \in M$ is called a limit point of $A$ if every neighborhood of $x$ contains a point of $A$ that is different from $x$ itself, that is, if $(B_\varepsilon(x) \setminus \{x\}) \cap A \neq \emptyset$ for every $\varepsilon > 0$. If $x$ is a limit point of $A$, show that every neighborhood of $x$ contains infinitely many point of $A$.

   Solution:

5. (Car. 4.48) A metric space is called separable if it contains a countable dense subset. Find example of countable dense sets in $\mathbb{R}$, in $\mathbb{R}^2$, and in $\mathbb{R}^n$.

   Solution:
6. (Car. 5.17) Let \( f, g : (M, d) \to (N, \rho) \) be continuous, and let \( D \) be a dense subset of \( M \). If \( f(x) = g(x) \) for all \( x \in D \), show that \( f(x) = g(x) \) for all \( x \in M \). If \( f \) is onto, show that \( f(D) \) is dense in \( N \).

Solution:

7. (Car. 5.20) If \( d \) a metric on \( M \), show that \(|d(x, z) - d(y, z)| \leq d(x, y)\) and conclude that the function \( f(x) = d(x, z) \) is continuous on \( M \) for any fixed \( z \in M \). This says that \( d(x, y) \) is separately continuous - continuous in each variable separately.

Solution:

8. (Car. 5.25) A function \( f : (M, d) \to (N, \rho) \) is called Lipschitz if there is a constant \( K < \infty \) such that \( \rho(f(x), f(y)) \leq Kd(x, y) \) for all \( x, y \in M \). Prove that a Lipschitz mapping is continuous.

Solution:

9. (Car. 5.36) Suppose that we are given a point \( x \) and a sequence \( (x_n) \) in a metric space \( M \), and suppose that \( f(x_n) \to f(x) \) for every continuous, real-valued function \( f \) on \( M \). Does it follow that \( x_n \to x \) in \( M \)? Explain.

Solution:

10. (Car. 7.5) Prove that \( A \) is totally bounded if and only if \( \bar{A} \) is totally bounded.

Solution:
11. (Car. 7.10) Prove that a totally bounded metric space $M$ is separable. [Hint: For each $n$, let $D_n$ be a finite $(1/n)$-net for $M$. Show that $D = \bigcup_{n=1}^\infty D_n$ is a countable dense set.]

**Solution:**

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**Optional Exercises**

12. (Car. 4.6 -optional) Give an example of an infinite closed set in $\mathbb{R}$ containing only irrationals. Is there an open set consisting entirely of irrationals?

**Solution:**

13. (Car. 4.7 -optional) Show that every open set in $\mathbb{R}$ is the union of (countably many) open intervals with rational endpoints. Use this to show that the collection $\mathcal{U}$ of all open subsets of $\mathbb{R}$ has the same cardinality as $\mathbb{R}$ itself.

**Solution:**

14. (Car. 4.14 -optional) Show that the set $A = \{ x \in \ell_2 : |x_n| \leq 1/n, \ n = 1, 2, \ldots \}$ is a closed set in $\ell_2$ but that $B = \{ x \in \ell_2 : |x_n| < 1/n, \ n = 1, 2, \ldots \}$ is not an open set. [Hint: Does $B \supset B_\varepsilon(0)$?]

**Solution:**

15. (Car. 4.34 -optional) Show that $x$ is a limit point of $A$ if and only if there is a sequence $(x_n)$ in $A$ such that $x_n \to x$ and $x_n \neq x$ for all $n$.

**Solution:**
16. (Car. 4.46 -optional) A set $A$ is said to be **dense** in $M$ (or, as some authors say, **everywhere dense**) if $\bar{A} = M$. For example, both $\mathbb{Q}$ and $\mathbb{R}\setminus\mathbb{Q}$ are dense in $\mathbb{R}$. Show that $A$ is dense in $M$ if and only if any of the following hold:

(a) Every point in $M$ is the limit of a sequence from $A$.
(b) $B_\varepsilon(x) \cap A \neq \emptyset$ for every $x \in M$ and every $\varepsilon > 0$.
(c) $U \cap A \neq \emptyset$ for every nonempty open set $U$.
(d) $A^c$ has empty interior.

**Solution:**

17. (Car. 4.58 -optional) Let $(r_n)$ be an enumeration of $\mathbb{Q}$. For each $n$, let $I_n$ be the open interval centered at $r_n$ of radius $2^{-n}$, and let $U = \bigcup_{n=1}^{\infty} I_n$. Prove that $U$ is a proper, open, dense subset of $\mathbb{R}$ and that $U^c$ is nowhere dense in $\mathbb{R}$.

**Solution:**

18. (Car. 5.30 -optional) Let $f : (M,d) \to (N,\rho)$. Prove that $f$ is continuous if and only if $f(\bar{A}) \subset \overline{f(A)}$ for every $A \subset M$ if and only if $f^{-1}(B^c) \subset \overline{f^{-1}(B)}$ for every $B \subset N$. Give an example of a continuous $f$ such that $f(\bar{A}) \neq \overline{f(A)}$ for some $A \subset M$.

**Solution:**

19. (Car. 5.42 -optional) Suppose that $f : \mathbb{Q} \to \mathbb{R}$ is Lipschitz. Show that $f$ extends to a continuous function $h : \mathbb{R} \to \mathbb{R}$. Is $h$ unique? Explain. [Hint: Given $x \in \mathbb{R}$, choose a sequence of rationals $(r_n)$ converging to $x$ and argue that $h(x) = \lim_{n \to \infty} f(r_n)$ exists and is actually independent of the sequence $(r_n)$.] 

**Solution:**

20. (Car. 7.9 -optional) Give an example of a closed bounded subset of $\ell_\infty$ that is not totally bounded.

**Solution:**